Parallel Computing on Solving Pennes Bioheat Equation

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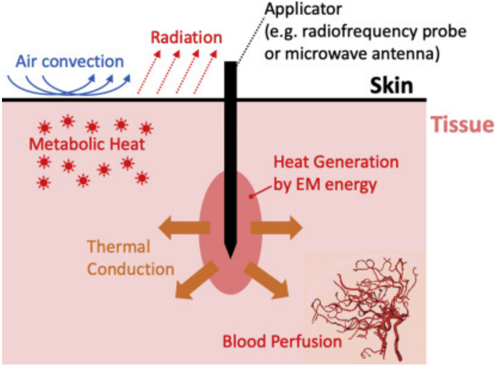
Pennes Experiment

Pennes goal was to evaluate the applicability of heat flow theory to the forearm in basic terms of local rate of tissue heat production and volume flow of blood



Pennes Assumptions

- The cross-section of a forearm is cylindrical
- The rate of heat production by tissue will be considered uniform throughout the forearm
- The volume flow of blood is constant
- The specific thermal conductivity K will be uniform

Pennes Bioheat Equation

$$cp\frac{\partial T}{\partial t} = -K\left[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + Q_m + Q_b \qquad (1.1)$$

c	Coefficient of heat for tissue (J / kg $\cdot°C$)	p	Density of tissue (kg/m^3)
K	Specific thermal conductivity of tissue (Watts / m $\cdot°C$)	T	Tissue temperature ($°C$)
Q_m	Rate of tissue heat production (Watts/m^3)	Q_b	Rate of heat transfer, blood to tissue (Watts/m^3)

Steady State Equation

The steady state of the equation was used to simplify the calculations. The steady state means: $\frac{\partial T}{\partial t} = 0$ from the original equation.
The general equation can then be formulated as:

$$k\nabla^2 T + Q_{met} - Q_{blood} = 0 \tag{1.2}$$

where:

- $\nabla^2 T$ is the Laplacian operator representing diffusion of heat.

Updating Formula

From equation 1.2, I descretized it using the finite difference method. This resulted in the following updating formula:

$$T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - \frac{q \cdot dx^2}{k}\right) \quad (2.1)$$

- q is $Q_{met} - Q_{blood}$, the heat generation rate (W/m^3)
- dx^2 is the mesh size, the length or spacing between the discrete points

Applying Jacobi to the Temperature Update Equation

To apply the Jacobi method, we introduce an iteration index k and rewrite the equation as:

$$T_{i,j}^{(k+1)} = \frac{1}{4}\left(T_{i+1,j}^{(k)} + T_{i-1,j}^{(k)} + T_{i,j+1}^{(k)} + T_{i,j-1}^{(k)} - \frac{q\,dx^2}{k}\right)$$

Procedure:

- Start with an initial guess $T_{i,j}^{(0)}$ for every grid point.
- Update the temperature at each grid point using the above iterative formula.
- Repeat the process until the solution converges (i.e., the change in $T_{i,j}$ becomes negligibly small).

Jacobi Results

Running and timing the Jacobi code, we get the following results:

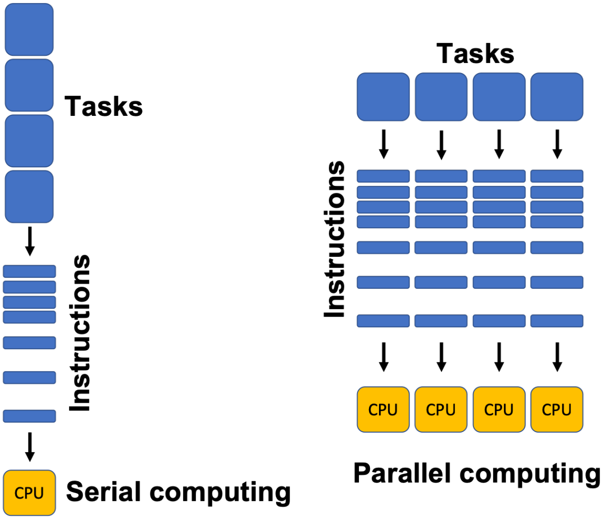
- Completed in 24,441 iterations with an error of 9.99666781353880 65E-07
- Run time: 5.044s

Changing parameters and number of grid points can significantly alter the time it takes to converge. This process can be quite lengthy depending on the problem parameters.

What is Parallel Computing?

- Breaking a big problem into smaller tasks.
- Solving those tasks simultaneously using multiple processors.
- Faster and more efficient than doing one task at a time.
- To speed up computation, we employ parallel computing using the Message Passing Interface (MPI).
- The parallel method used for my research is Spatial Domain Decomposition.

Parallel Computing Visualization



Difference in Computational Time

Method	Number of CPU's	Run Time
Jacobi	1	5.044s
Parallel Jacobi	2	0.798s
Parallel Jacobi	4	0.645s
Parallel Jacobi	5	0.598s
Parallel Jacobi	6	0.700s
Parallel Jacobi	8	0.732s
Parallel Jacobi	10	0.871s

Table: Comparison of Jacobi implementations

As we can see, more processors does not mean faster and faster solving time. This is due to Amdahl's Law.

The Parareal Algorithm

The next goal of the research was to apply the Parareal Algorithm to the Pennes Bioheat Equation. For this, we go back to the original Partial Differential Equation:

$$cp\frac{\partial T}{\partial t} = -K\left[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + Q_m + Q_b \tag{5.1}$$

- This method was developed by Lions, Maday, and Turinici
- A method to speed up solving time-dependent problems.
- Much more complex since we are working with a partial differential equation

Parareal Algorithm: Core Concepts

1. Coarse Propagator $u_{n+1}^{(k)} = G(u_n^{(k)}, t_n, \Delta t)$

- Cheaper, faster, but less accurate
- Runs sequentially over the time domain

2. Fine Propagator $u_{n+1}^{(k)} = F(u_n^{(k-1)}, t_n, \Delta t)$

- Computationally expensive
- Can be run in parallel over subdomains

3. Predictor-Corrector Update Rule

$$u_{n+1}^{(k+1)} = G(u_n^{(k+1)}, t_n, \Delta t) + F(u_n^{(k)}) - G(u_n^{(k)})$$

- Combines fast parallel fine step with cheap coarse prediction
- Iterated until convergence

Parareal Algorithm Output

Iteration	Max Difference
1	1.06811523E-04
2	8.77380371E-05
3	8.01086426E-05
4	7.87016254E-05
5	7.67235015E-05
6	7.24560251E-05
7	7.02351095E-05
8	6.67041494E-05
9	6.23001457E-05
10	5.72204590E-06

Table: Parareal Iterations and Maximum Differences(between each iteration) in 1D, Runtime: 0.084s

Parareal Results

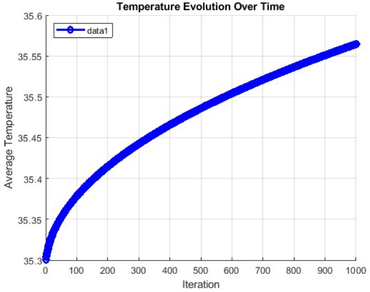


Figure: Spatial Domain Decomposition on the Ordinary Differential Equation

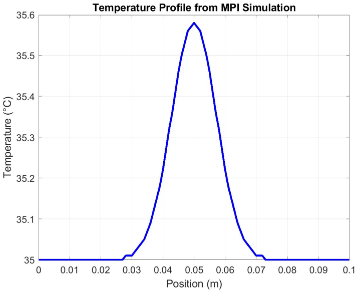


Figure: Parareal Algorithm on the Partial Differential Equation

Conclusion

- The Spatial Domain Decomposition and Parareal Algorithm solve the problem much more efficiently compared to the sequential method.
- Ultimately, the goal is to combine these methods to increase efficiency even more
- Additionally, further research is needed to apply the Parareal Algorithm in the 2D and 3D case

References

1. Lions, Jacques-Louis, et al. "Résolution d'edp par un schéma en temps pararéel." Comptes Rendus de l'Académie Des Sciences - Series I - Mathematics, vol. 332, no. 7, Apr. 2001, pp. 661–668.
2. Pennes, Harry H. "Analysis of Tissue and Arterial Blood Temperatures in the Resting Human Forearm." Journal of Applied Physiology, vol. 1, no. 2, Aug. 1948, pp. 93–122
3. Staff, Gunnar A. The Parareal Algorithm: A Survey of Present Work. Norwegian University of Science and Technology, Dept. of Mathematical Sciences, Summer 2003.
4. Wissler, Eugene H. "Pennes' 1948 paper revisited." Journal of Applied Physiology, vol. 85, no. 1, 1 July 1998, pp. 35–41.
5. Yue, Kai, et al. "An Analytic Solution of One-dimensional Steady-state Pennes' Bioheat Transfer Equation in Cylindrical Coordinates." Journal of Thermal Science, vol. 13, no. 3, Aug. 2004, pp. 255–258.