DISSECTIONS OF LACUNARY ETA QUOTIENTS AND IDENTICALLY VANISHING COEFFICIENTS

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ABSTRACT. For any function $A(q) = \sum_{n=0}^{\infty} a_n q^n$ define

$$A_{(0)} := \{n \in \mathbb{N} : a_n = 0\}.$$

Now suppose $C(q)$ and $D(q)$ are two functions whose m-dissections are given by

$$C(q) = c_0 G_0(q^m) + c_1 q G_1(q^m) + \cdots + c_{m-1} q^{m-1} G_{m-1}(q^m),$$
$$D(q) = d_0 G_0(q^m) + d_1 q G_1(q^m) + \cdots + d_{m-1} q^{m-1} G_{m-1}(q^m).$$

If it is the case that $c_i = 0 \iff d_i = 0$, $i = 0, 1, \ldots, m-1$, then we say that $C(q)$ and $D(q)$ have *similar m-dissections*, and then it is also clear that $C_{(0)} = D_{(0)}$, in which case we say that $C(q)$ and $D(q)$ have *identically vanishing coefficients*.

In the present paper some new 4-dissections of particular eta quotients are developed. These are used in conjunction with known 2- and 3-dissections to prove many results on the identical vanishing of coefficients in sets of 4-, 6- and 8- lacunary eta quotients, results which found experimentally and partially proved in another paper by the present authors.

Similar arguments allow many results of the form $C_{(0)} \subsetneqq D_{(0)}$ to be proved for many pairs of lacunary eta quotients $C(q)$ and $D(q)$.

1. INTRODUCTION

The work in the present paper continues to examine the phenomena described in a previous paper of the authors [12], which was itself a continuation of work that began in [11], which in turn was motivated by a result of Han and Ono in [3].

The result of Han and Ono is recast in the next Theorem.

Theorem 1.1. (*Han and Ono,* [3, Theorem 1.4, page 307]) *Define the sequences* $\{a_n\}$ *and* $\{b_n\}$ *by*

$$f_1^8 =: \sum_{n=0}^{\infty} a_n q^n, \qquad \frac{f_3^3}{f_1} =: \sum_{n=0}^{\infty} b_n q^n, \qquad \text{where } f_i := \prod_{n=1}^{\infty}(1-q^{in}), \quad i \in \mathbb{Z}^+. \tag{1.0.1}$$

Then

$$a_n = 0 \iff b_n = 0. \tag{1.0.2}$$

Moreover, we have that $a_n = b_n = 0$ *precisely for those non-negative* n *for which* $ord_p(3n+1)$ *is odd for some prime* $p \equiv 2 \pmod 3$.

Their result motivated the work in [11], where the present authors investigated if a similar situation held for other pairs of eta quotients (an *eta quotient* being a finite product of the form $\prod_j f_j^{n_j}$, for some $j \in \mathbb{N}$ and some $n_j \in \mathbb{Z}$, with a product with all $n_j > 0$ being termed an *eta product*). One example result from that paper is the following:

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If the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are defined by

$$f_1^4 =: \sum_{n=0}^{\infty} a_n q^n, \qquad \frac{f_1^8}{f_2^2} =: \sum_{n=0}^{\infty} b_n q^n, \qquad \frac{f_1^{10}}{f_3^2} =: \sum_{n=0}^{\infty} c_n q^n,$$

then

$$a_n = 0 \iff b_n = 0 \iff c_n = 0,$$

with the criterion for $a_n = b_n = c_n = 0$ being that of Serre [17, page 210].

These investigations were continued in [12], where the results of extensive experimental searches were described, and it was found that the phenomenon was quite common. To discuss this work further, we introduce some notation. For a function $A(q) = \sum_{n\geq 0} a_n q^n$ we write

$$A_{(0)} := \{n \in \mathbb{N} : a_n = 0\}$$

If $A(q)$ and $B(q)$ are two functions for which $A_{(0)} = B_{(0)}$, then for ease of discussion, we say that *the coefficients vanish identically*, or that $A(q)$ and $B(q)$ have *identically vanishing coefficients*. If $A_{(0)} \subseteq B_{(0)}$, we say that $A(q)$ has *vanishing behavior similar to* $B(q)$.

It was found that if $A(q)$ is any one of f_1^r, $r = 4, 6, 8, 10, 14$ and 26 (lacunary eta quotients whose vanishing coefficient behaviour was described by Serre [17]) or $f_1^3 f_2^3$ (the simplest case of an infinite family of lacunary eta quotients stated by Ono and Robins [14, page 1027]), then in each case there were a large numbers of eta quotients $B(q)$ such that $A_{(0)} = B_{(0)}$. Further, in each case there were also many other eta quotients $C(q)$ such that $A_{(0)} \subsetneqq C_{(0)}$.

To illustrate the full complexity of the situation, we examine the case of f_1^4 in more detail. Our limited search (see [12] for details about the extent of this experimental search) found a total of 72 eta quotients $B(q)$ for which it appeared $f^4_{1\,(0)} = B_{(0)}$. In addition, this search found 78 additional eta quotients with the property that for each such eta quotient $C(q)$, it seemed $f^4_{1\,(0)} \subsetneqq C_{(0)}$. Moreover, it appears that all 150 eta quotients $B(q)$ may be organized into 19 collections (labelled I - XIX in what follows) in a tree-like structure by partially ordering the corresponding $B_{(0)}$ by inclusion.

Table 1: Eta quotients with vanishing behaviour similar to f_1^4

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	72	II *	4
III †	2	IV	6
V †	2	VI *	4
VII *	6	VIII *	8
IX *	4	X	4
XI	14	XII †	2
XIII †	2	XIV †	2
XV	4	XVI †	2
XVII	4	XVIII †	2
XIX †	6		

Thus, for example, all 14 eta quotients in the collection labelled XI, where

$$XI = \left\{ \frac{f_2 f_8^{14} f_{12}^2}{f_4^6 f_6 f_{16}^5 f_{24}}, \frac{f_6 f_8^{13}}{f_2 f_4^3 f_{12} f_{16}^5}, \frac{f_2^2 f_8 f_{12}^2}{f_4^2 f_{24}}, \frac{f_8^{11}}{f_2^2 f_{16}^5}, \frac{f_4^4 f_{12}^2}{f_2^2 f_8 f_{24}}, \frac{f_2^2 f_8^{13}}{f_4^6 f_{16}^5}, \frac{f_4^{15} f_6 f_{24}}{f_2^5 f_8^5 f_{12}^3}, \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

appeared to have identically vanishing coefficients. Likewise for any other pair of eta quotients that both lie in any of the other collections. Collection I is the collection containing f_1^4. An asterisk symbol * next to a group label in Table 1 indicates that it is proven in the present paper that all eta quotients in the corresponding group have identically vanishing coefficients. A dagger symbol † next to a group label in Table 1 indicates either that the group members trivially have identically vanishing coefficients (because the group contains just two members, one being the $q \to -q$ partner of the other) or that it was shown by the present authors in [12] that all eta quotients in the group have identically vanishing coefficients.

The relationships between eta quotients in different collections is illustrated in Figure 1.

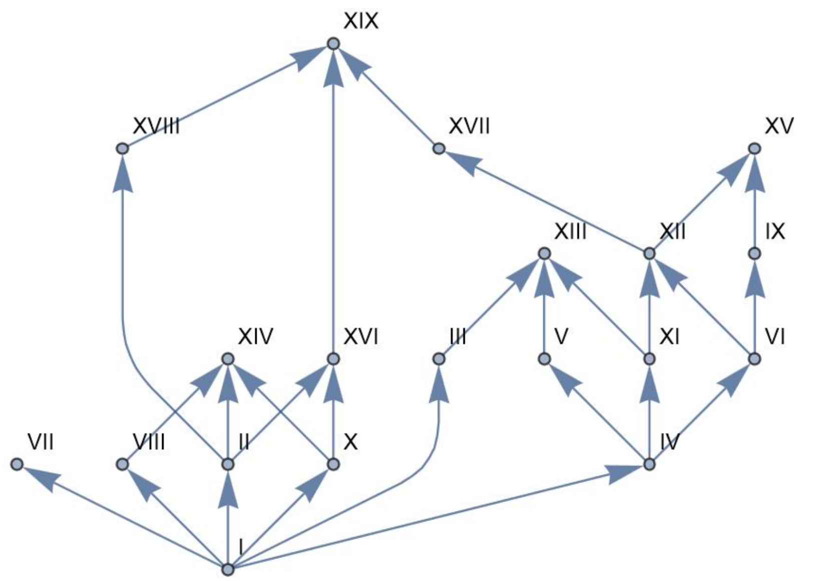

FIGURE 1. The grouping of eta-quotients in Table 1, which have vanishing coefficient behaviour similar to f_1^4

Thus the arrow from VIII to XIV indicates that if $A(q)$ is any of the 8 eta quotients in collection VIII and $B(q)$ is either of the 2 eta quotients in collection XIV, where

$$VIII = \left\{ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}} \right\},$$

$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\},$$

then $A_{(0)} \subsetneq B_{(0)}$. A similar meaning for any other arrow in this figure is to be understood. The inclusion just mentioned, between groups VIII and XIV, is one of several such inclusion results indicated by the arrows in Figure 1 that are proven in the present paper.

We likewise summarize what experiment suggests about the collections of eta quotients with vanishing coefficient behaviour similar to f_1^6 in the following table and graph.

Table 2: Eta quotients with vanishing behaviour similar to f_1^6

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	42	II *	4
III *	4	IV	16
V †	2	VI †	2
VII *	4	VIII *	4
IX *	4	X	10
XI †	2	XII *	4
XIII *	8	XIV *	4
XV	8	XVI †	2
XVII	8	XVIII †	2
XIX †	2	XX †	2
XXI *	4	XXII *	6
XXIII †	2	XXIV *	4
XXV *	4	XXVI	4
XXVII †	2	XXVIII †	6
XXIX †	6		

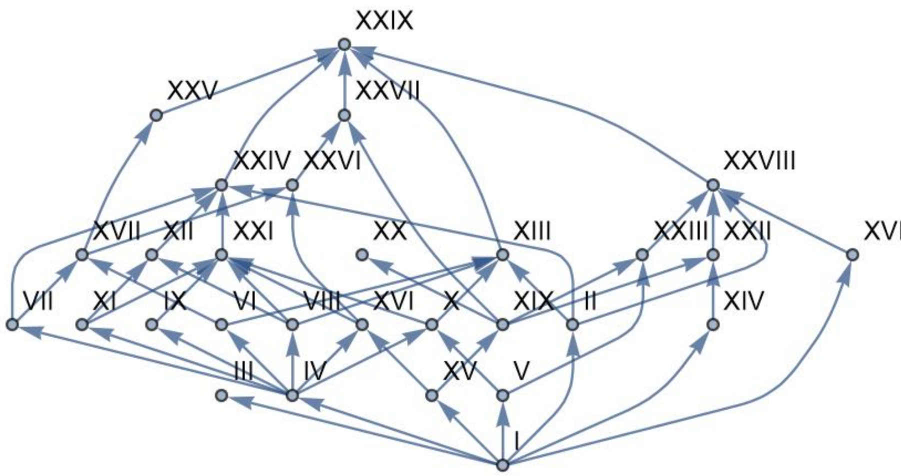

FIGURE 2. The grouping of eta-quotients in Table 2, which have vanishing coefficient behaviour similar to f_1^6

As with Table 1, all eta quotients in the same collection (tagged with a Roman numeral) in Table 2 appear to have identically vanishing coefficients, and the arrows in Figure 2 have the same meaning as they did in Figure 1. For example, the arrow from VIII to XXI indicates that if $A(q)$ is any of the 4 eta quotients in collection VIII and $B(q)$ is any of the 4 eta quotients in collection XXI, where

$$VIII = \left\{ \frac{f_2^5 f_3^2 f_{12}^{11}}{f_1^2 f_4^2 f_6^6 f_{24}^4}, \frac{f_1^2 f_{12}^9}{f_2 f_3^2 f_{24}^4}, \frac{f_2^2 f_{12}^9}{f_1 f_3 f_6 f_{24}^4}, \frac{f_1 f_3 f_4 f_{12}^{10}}{f_2 f_6^4 f_{24}^4} \right\},$$

$$XXI = \left\{ \frac{f_8^2 f_{12}^{14}}{f_4 f_6^5 f_{24}^6}, \frac{f_6^5 f_8^2}{f_4 f_{12} f_{24}}, \frac{f_4^2 f_{12}^{13}}{f_6^5 f_8 f_{24}^5}, \frac{f_4^2 f_6^5}{f_8 f_{12}^2} \right\},$$

then $A_{(0)} \subsetneqq B_{(0)}$. Note that collection I is the one containing f_1^6. The * and † symbols in Table 2 likewise have the same meaning as described above for their use in Table 1. In particular, a * symbol next to a group label in Table 2 here also indicates that it is proven in the present paper that all eta quotients in the corresponding group have identically vanishing coefficients.

A smaller number of results were derived for eta quotients with vanishing coefficient behaviour similar to f_1^r, $r \in \{8, 10, 14, 26\}$ and $f_1^3 f_2^3$. These are described later in the paper, where the tables and diagrams are to be understood as above, with the arrows and the * and † symbols having the same meaning as in this section.

We did not prove all of the hundreds of results on the vanishing of eta quotient coefficients suggested by experiment and outlined in [12]. We described several general methods in [12, Section 8] that enable results of the form $A_{(0)} = B_{(0)}$ or $A_{(0)} \subsetneqq B_{(0)}$ to be proved for eta quotients $A(q)$ and $B(q)$, and did prove a large number of these by way of illustration of these methods. For example, using the equivalent relation between the lacunarity and CM'ness of a cusp form found by Serre [16], as well as Ribet's characterization [15] of a CM newform, we were able to prove that $f_1^4{}_{(0)} \subseteq B_{(0)}$ for any $B(q)$ of the 150 eta quotients whose coefficients are experimentally supposed to have vanishing behaviour similar to that of f_1^4, and similarly, $f_1^6{}_{(0)} \subseteq B_{(0)}$ for any eta quotient $B(q)$ obtained in the search associated with f_1^6. In particular, we showed that $f_1^4{}_{(0)} \subsetneqq B_{(0)}$ for any eta quotient $B(q)$ in the aforementioned collections XI, VIII, or XIV associated with Table 1, and $f_1^6{}_{(0)} \subsetneqq B_{(0)}$ for any eta quotient $B(q)$ in the aforementioned collections VIII or XXI associated with Table 2.

This present paper concerns a method that was not used to any great extent in [12], but which can also be used to derive results of the form $A_{(0)} = B_{(0)}$ or $A_{(0)} \subsetneqq B_{(0)}$ for eta quotients $A(q)$ and $B(q)$. This method involves constructing m-dissections of eta quotients whose dissection components differ by a scalar. We define the m-dissection of a power series next.

Definition 1. By the m-dissection of a function $G(q) = \sum_{n=0}^{\infty} g_n q^n$ we mean an expansion of the form

$$G(q) = \gamma_0 G_0(q^m) + \gamma_1 q G_1(q^m) + \cdots + \gamma_{m-1} q^{m-1} G_{m-1}(q^m), \tag{1.0.3}$$

where each dissection component $G_i(q^m)$ is not identically zero ($\gamma_i = 0$ is allowed). In other words, for each i, $0 \le i \le m-1$,

$$\gamma_i G_i(q^m) = \sum_{n=0}^{\infty} g_{mn+i} q^{mn}. \tag{1.0.4}$$

Now suppose $C(q)$ and $D(q)$ are two functions whose m-dissections are given by

$$\begin{aligned} C(q) &= c_0 G_0(q^m) + c_1 q G_1(q^m) + \cdots + c_{m-1} q^{m-1} G_{m-1}(q^m), \\ D(q) &= d_0 G_0(q^m) + d_1 q G_1(q^m) + \cdots + d_{m-1} q^{m-1} G_{m-1}(q^m). \end{aligned} \tag{1.0.5}$$

There are two cases of interest.

1) Suppose that $c_i = 0 \Longleftrightarrow d_i = 0$, $i = 0, 1, \ldots, m-1$, and then it is clear that $C_{(0)} = D_{(0)}$. If the c_1, d_i satisfy the condition just stated, we say that $C(q)$ and $D(q)$ have *similar* m-dissections.

2) On the other hand, if $c_j \neq 0$ and $d_j = 0$ for one or more $j \in \{0, 1, \ldots, m-1\}$ and otherwise $c_i = 0 \Longleftrightarrow D_i = 0$, then $C_{(0)} \subsetneqq D_{(0)}$.

As will be shown, many pairs of eta quotients $(C(q), D(q))$ have m-dissections (here m is usually a small integer such as 2, 3 or 4) that are related in one of the two ways just described, thus allowing one of the two conclusions about vanishing coefficients to be drawn.

We remark in passing that in what follows, sometimes there may be more than one way to arrive at, and express, the m-dissection of an eta quotient, in which case equating corresponding components may possibly lead to some interesting identities.

Remark: Before proceeding we note the following. Suppose $A(q)$, $B(q)$, $C(q)$ and $D(q)$ are eta quotients and $m > 1$ is a positive integer. Let $A'(q) := A(q^m)$, $B'(q) := B(q^m)$, $C'(q) := C(q^m)$ and $D'(q) := D(q^m)$. Then it is easy to see that

$$A_{(0)} = B_{(0)} \Longleftrightarrow A'_{(0)} = B'_{(0)},$$
$$C_{(0)} \subset D_{(0)} \Longleftrightarrow C'_{(0)} \subset D'_{(0)}. \tag{1.0.6}$$

What this means is that some inclusion and equality results for groups of eta quotients in Table 2 and Figure 2 (recall that these represent groups of eta quotients with vanishing coefficient behaviour similar to f_1^6) derive trivially from such results holding in the corresponding tables for, respectively, f_1^2 and f_1^3 through, respectively, the dilations $q \to q^3$ and $q \to q^2$ (of course a similar statement may be made about the table for f_1 and the dilation $q \to q^6$). See [12] for more on this.

Further, if Table 1 and Figure 1 in their entirety represent the true situation for eta quotients with vanishing coefficient behaviour similar to f_1^4, then this table and graph are embedded in their entirety, via the dilation $q \to q^2$, in the corresponding table and graph for f_1^8, namely Table 3 and Figure 3 below. We return to this in subsection 3.3.

2. Some elementary dissection results

A fundamental result is the Jacobi triple product identity,

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n^2} = (zq, q/z, q^2; q^2)_\infty, \tag{2.0.1}$$

stated by Jacobi [13], but first proved by Gauss [2]. The identities in the Lemma 2.1 are special cases of this identity.

Lemma 2.1.

$$\frac{f_2^5}{f_1^2 f_4^2} = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.0.2}$$

$$\frac{f_1^2}{f_2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \tag{2.0.3}$$

The expansions in Lemma 2.1 are fundamental in the derivation of m-dissections of eta quotients, and easily lead to the first two 2-dissections in Lemma 2.2, for example.

We often make the substitution $q \to -q$ in an eta quotient but wish to write the resulting product also as an eta quotient. This leads to the following frequently employed identity:

$$(-q; -q)_\infty = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty (q^4; q^4)_\infty} = \frac{f_2^3}{f_1 f_4} \tag{2.0.4}$$

If $g(q) = f(-q)$, for simplicity we will call $g(q)$ the "$q \to -q$ partner" of $f(q)$. The relevance in the present context is that a function and its $q \to -q$ partner have identically vanishing coefficients.

2.1. **2-Dissections.**

Lemma 2.2. *The following 2-dissections hold:*

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q \frac{f_{16}^2}{f_8}, \tag{2.1.1}$$

$$\frac{f_1^2}{f_2} = \frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8}, \tag{2.1.2}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.1.3}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^3 f_6^3}{f_2^9 f_{12}}\left(\frac{f_4^3}{f_{12}} + 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right), \tag{2.1.4}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.1.5}$$

$$\frac{f_1}{f_3^3} = \frac{f_2^3 f_{12}^3}{f_4 f_6^9}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} - q\frac{f_{12}^3}{f_4}\right), \tag{2.1.6}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{2.1.7}$$

$$\frac{1}{f_1 f_3} = \frac{f_4 f_{12}}{f_2^3 f_6^3}\left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} + q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}\right), \tag{2.1.8}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{2.1.9}$$

$$\frac{1}{f_1^4} = \frac{f_4^4}{f_2^{12}}\left(\frac{f_4^{10}}{f_2^2 f_8^4} + 4q\frac{f_2^2 f_8^4}{f_4^2}\right), \tag{2.1.10}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}, \tag{2.1.11}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6^3}{f_2^3 f_{12}}\left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} + q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}\right), \tag{2.1.12}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}, \tag{2.1.13}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^2 f_6^6}{f_2^6 f_{12}^2}\left(\frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} + 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}\right). \tag{2.1.14}$$

Proof. The first two of these follow from separating the corresponding series expansion at (2.0.2) and (2.0.3), respectively, into sums over odd and even indices n, and then using the Jacobi triple product identity (2.0.1) identity to convert the resulting sums back into infinite products.

Identity (2.1.3) is Equation (1.35) in [8], and (2.1.4) is derived from (2.1.3) upon the replacement $q \to -q$.

Identities (2.1.5) and (2.1.6) are stated in Lemma 2.1 of [9], identity (2.1.8) is stated in Lemma 2.4 of [9], and (2.1.7) is derived from the latter identity after replacing q with $-q$. Finally, (2.1.10) is stated in Lemma 2.3 of [9], and (2.1.9) follows from (2.1.10) once again upon replacing q with $-q$.

All four of the 2-dissections (2.1.11) - (2.1.14) may be found in [7, Section 30.10]. □

Trivially, the series expansions of eta quotients $A(q)$ and $A(-q)$ have identically vanishing coefficients, so we would like to develop less trivial criteria. The first of these that we have is contained in the following lemma. Although still elementary, it does provide an explanation for many pairs of eta quotients with identically vanishing coefficients.

Lemma 2.3. *Suppose $A(q)$ and $B(q)$ are functions with the indicated 2-dissections.*

$$\begin{aligned} A(q) &= A_0(q^2) + qA_1(q^2), \\ B(q) &= B_0(q^2) + qB_1(q^2). \end{aligned} \tag{2.1.15}$$

Suppose

$$A_0(q^2)B_1(q^2) = \kappa A_1(q^2)B_0(q^2), \tag{2.1.16}$$

for some non-zero real number κ. *For any even function* $C(q^2)$, *define the pair of functions* $F(q)$. $G(q)$ *by*

$$F(q) := A(q)C(q^2), \qquad G(q) := B(q)\frac{A_0(q^2)}{B_0(q^2)}C(q^2). \tag{2.1.17}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.1.18}$$

Proof. From (2.1.15) and (2.1.16),

$$\begin{aligned} A(q)B_0(q^2)C(q^2) &= A_0(q^2)B_0(q^2)C(q^2) + qA_1(q^2)B_0(q^2)C(q^2), \\ B(q)A_0(q^2)C(q^2) &= B_0(q^2)A_0(q^2)C(q^2) + qB_1(q^2)A_0(q^2)C(q^2) \\ &= A_0(q^2)B_0(q^2)C(q^2) + \kappa qA_1(q^2)B_0(q^2)C(q^2), \end{aligned} \tag{2.1.19}$$

and thus both products on the left sides at (2.1.19) have identical 2-dissections, up to the factor κ, and hence have identically vanishing coefficients. The result at (2.1.18) follows upon making the replacement $C(q^2) \to C(q^2)/B_0(q^2)$. □

Theorem 2.1. *Let* $C(q^2)$ *be any even eta quotient. Let* $F(q)$ *and* $G(q)$ *be any pair of eta quotients in the following list:*

$$\left\{ \frac{f_3}{f_1^3}C(q^2),\ \frac{f_1^3 f_4^3 f_6^3}{f_2^9 f_3 f_{12}}C(q^2),\ \frac{f_3^3}{f_1}\frac{f_4^3 f_6}{f_2^7 f_{12}}C(q^2),\ \frac{f_1}{f_3^3}\frac{f_4^4 f_6^{10}}{f_2^{10} f_{12}^4}C(q^2) \right\}. \tag{2.1.20}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.1.21}$$

Proof. The claim follows for the first and second eta quotient, since the second is derived from the first upon replacing q with $-q$ and using (2.0.4).

In Lemma 2.3, let

$$A(q) = \frac{f_3}{f_1^3}, \qquad B(q) = \frac{f_3^3}{f_1},$$

so that (2.1.16) holds (with $\kappa = 3$) by (2.1.4) and (2.1.5), thus proving the claim for the first and third eta quotient. The claim for the fourth eta quotient follows upon replacing q with $-q$ in the third eta quotient and again using (2.0.4). □

2.2. **3-Dissections.**

Lemma 2.4. *The following 3-dissections hold.*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{2.2.1}$$

$$\frac{f_1 f_4}{f_2} = \frac{f_3 f_{12} f_{18}^5}{f_6^2 f_9^2 f_{36}^2} - q\frac{f_9 f_{36}}{f_{18}}, \tag{2.2.2}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.2.3}$$

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_{18}^5}{f_9^2 f_{36}^2} + \frac{2q f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}, \tag{2.2.4}$$

$$(2.2.5)\qquad \frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6},$$

$$(2.2.6)\qquad \frac{f_1^2 f_4^2}{f_2^5} = \frac{f_3^8 f_{12}^8 f_{18}^{15}}{f_6^{20} f_9^6 f_{36}^6} - \frac{2q f_3^7 f_{12}^7 f_{18}^9}{f_6^{18} f_9^3 f_{36}^3} + \frac{4q^2 f_3^6 f_{12}^6 f_{18}^3}{f_6^{16}}.$$

Proof. These are all known results or are easy consequences of known results. Identities (2.2.1) and (2.2.3) may be found in [7, page 132], for example, and identities (2.2.2) and (2.2.4) are the $q \to -q$ partners, respectively, of the former two identities.

The identity (2.2.5) was proven in [10, Theorem 1], and (2.2.6) is its $q \to -q$ partner. ☐

The next 3-dissection result needs some preliminary notation and results. Recall that the Borwein theta functions $a(q)$, $b(q)$ and $c(q)$ are defined by

$$(2.2.7)\qquad a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = \frac{f_2^5 f_6^5}{f_1^2 f_3^2 f_4^2 f_{12}^2} + 4q\frac{f_4^2 f_{12}^2}{f_2 f_6},$$

$$b(q) = \omega^{n-m} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = \frac{f_1^3}{f_3},$$

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2} = 3q^{1/3}\frac{f_3^3}{f_1},$$

where $\omega = \exp(2\pi i/3)$. The infinite product representations for $a(q)$, $b(q)$ and $c(q)$ were proven by the authors in [1].

Note also that the 2-dissections of the infinite products associated with $b(q)$ and $c(q)$, namely f_1^3/f_3 and f_3^3/f_1, are given in Lemma 2.2, and these 2-dissections are used elsewhere, for example in Lemma 2.11, to prove other dissection results.

Lemma 2.5. *The following 3-dissections hold.*

$$(2.2.8)\qquad f_1^3 = a(q^3) f_3 - 3q f_9^3,$$

$$(2.2.9)\qquad \frac{1}{f_1^3} = \frac{f_9^3}{f_3^{10}}\left(a(q^3)^2 + 3q\frac{f_9^3}{f_3}a(q^3) + 9q^2\frac{f_9^6}{f_3^2}\right).$$

Proof. The identity at (2.2.8) was proven in [8], where it was stated as a 3-dissection of $b(q)$. The result (2.2.9) was proven in [6], where it was given as a 3-dissection of $1/b(q)$. ☐

Theorem 2.2. *Let $C(q^3)$ be any eta quotient whose series expansion contains only powers of q^3. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:*

$$(2.2.10)\qquad \left\{\frac{f_2^2}{f_1}C(q^3), \frac{f_1 f_4}{f_2}C(-q^3), \frac{f_1^2 f_6}{f_2 f_3}C(q^3), \frac{f_2^5 f_3 f_{12}}{f_1^2 f_4^2 f_6^2}C(-q^3)\right\},$$

$$(2.2.11)\qquad \left\{\frac{f_1^2 f_8}{f_4}C(q^3), \frac{f_2^6 f_8}{f_1^2 f_4^3}C(-q^3), \frac{f_3 f_4^5 f_{24}}{f_1 f_8^2 f_{12}^2}C(q^3), \frac{f_1 f_4^6 f_6^3 f_{24}}{f_2^3 f_3 f_8^2 f_{12}^3}C(-q^3)\right\},$$

$$(2.2.12)\qquad \left\{f_1^6 C(q^3), \frac{f_2^{18}}{f_1^6 f_4^6}C(-q^3), \frac{f_3^{12}}{f_1^3 f_9^3}C(q^3), \frac{f_1^3 f_4^3 f_6^{36} f_9^3 f_{36}^3}{f_2^9 f_3^{12} f_{12}^{12} f_{18}^9}C(-q^3)\right\},$$

Then

$$(2.2.13)\qquad F_{(0)} = G_{(0)}.$$

Proof. For (2.2.10), the result holds for the first and third eta quotients by (2.2.1) and (2.2.3). The claim holds more generally since the second and fourth eta quotients are the $q \to -q$ partners of the first and third, respectively.

For (2.2.11), the claims follow for the first and third eta quotients, by using, respectively, (2.2.3) combined with (2.2.2) (with q replaced with q^2), and (2.2.1) combined with (2.2.4) (with q replaced with q^2) to get

$$\frac{f_1^2 f_8}{f_4} = \frac{f_1^2}{f_2}\frac{f_2 f_8}{f_4} = \left(\frac{f_9^2}{f_{18}} - \frac{2q f_3 f_{18}^2}{f_6 f_9}\right)\left(\frac{f_6 f_{24} f_{36}^5}{f_{12}^2 f_{18}^2 f_{72}^2} - \frac{q^2 f_{18} f_{72}}{f_{36}}\right),$$
$$\frac{f_4^5}{f_1 f_8^2} = \frac{f_2^2}{f_1}\frac{f_4^5}{f_2^2 f_8^2} = \left(\frac{f_6 f_9^2}{f_3 f_{18}} + \frac{q f_{18}^2}{f_9}\right)\left(\frac{f_{36}^5}{f_{18}^2 f_{72}^2} + \frac{2q^2 f_{12}^2 f_{18} f_{72}}{f_6 f_{24} f_{36}}\right).$$

The claims in the theorem hold generally since the second and fourth eta quotients are the $q \to -q$ partners of the first and third, respectively.

For (2.2.12), the claim for the first and third eta quotients is an immediate consequence of (2.2.8) and (2.2.9), and once more holds generally as the second and fourth eta quotients are again the $q \to -q$ partners of the first and third, respectively.

□

Unless there is cancellation in the expansion of one of m-dissections, if $A(q)$ and $B(q)$ have identically vanishing coefficients, then trivially so have their squares. However it turns out that there is an additional eta quotient whose series expansion has coefficients that vanish identically with those of the squares of the eta quotients at (2.2.10).

Corollary 2.1. *Let $C(q^3)$ be any eta quotient with a power series expansion in q^3. Let $F(q)$ and $G(q)$ be any pair of eta quotients from the following list:*

(2.2.14)
$$\left\{\frac{f_2^4}{f_1^2}C(q^3),\ \frac{f_1^2 f_4^2}{f_2^2}C(-q^3),\ \frac{f_1^4 f_6^2}{f_2^2 f_3^2}C(q^3),\ \frac{f_2^{10} f_3^2 f_{12}^2}{f_1^4 f_4^4 f_6^4}C(-q^3),\ \frac{f_1 f_2 f_6}{f_3}C(q^3),\ \frac{f_3 f_2^4 f_{12}}{f_1 f_4 f_6^2}C(-q^3)\right\}.$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.2.15}$$

Proof. In light of the remark above and the fact that the sixth eta quotient in the list is the $q \to -q$ partner of the fifth, all that is necessary is to prove the statement for $F(q)$ equal to the first eta quotient in the list and $G(q)$ equal to the fifth. However, this is immediate from the facts that (2.2.1) and (2.2.3) give that

$$\frac{f_2^4}{f_1^2} = \frac{f_6^2 f_9^4}{f_3^2 f_{18}^2} + \frac{2q f_6 f_9 f_{18}}{f_3} + \frac{q^2 f_{18}^4}{f_9^2}, \tag{2.2.16}$$

$$\frac{f_1 f_2 f_6}{f_3} = \frac{f_1^2}{f_2}\frac{f_2^2}{f_1}\frac{f_6}{f_3} = \frac{f_6^2 f_9^4}{f_3^2 f_{18}^2} - \frac{q f_6 f_9 f_{18}}{f_3} - \frac{2q^2 f_{18}^4}{f_9^2}. \tag{2.2.17}$$

□

Another variation runs as follows. It can easily be seen from (2.2.1) and (2.2.3) that f_2^2/f_1 and $f_1^2 f_6/(f_2 f_3)$ have similar 3-dissections (where "similar" has the meaning assigned following (1.0.5)). The same is true if the dilation $q \to q^t$ is applied to this pair of eta quotients, where t is a positive integer, $t \not\equiv 0 \pmod 3$. We consider the particular cases $t = 2$ and $t = 4$ (to be used later) so that if $C(q^3)$ is any eta quotient with a power series expansion in q^3 and $F(q)$ and $G(q)$ are any pair of eta quotients from one of the following list,

$$\left\{\frac{f_4^2}{f_2}C(q^3), \frac{f_4^2}{f_2}C(-q^3), \frac{f_2^2 f_{12}}{f_4 f_6}C(q^3), \frac{f_2^2 f_{12}}{f_4 f_6}C(-q^3)\right\}, \tag{2.2.18}$$

$$\left\{\frac{f_8^2}{f_4}C(q^3),\ \frac{f_8^2}{f_4}C(-q^3),\ \frac{f_4^2 f_{24}}{f_8 f_{12}}C(q^3),\ \frac{f_4^2 f_{24}}{f_8 f_{12}}C(-q^3)\right\}, \tag{2.2.19}$$

then $F_{(0)} = G_{(0)}$.

2.3. **4-Dissections.** Remark: In follows, we sometimes leave a 4-dissection factored in the form

$$(A(q^4) + qB(q^4))(C(q^4) + q^2D(q^4)),$$

while still referring to it as the "4-dissection".

Before stating the next general vanishing coefficient result, we require the following preliminary results.

Lemma 2.6. *The following 4-dissections hold.*

$$f_1^2 f_2^7 = \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)^2, \tag{2.3.1}$$

$$\frac{1}{f_1^2 f_2^3} = \frac{f_8^8}{f_4^{22}}\left(\frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)^2. \tag{2.3.2}$$

Proof. For (2.3.1), write

$$f_1^2 f_2^7 = \frac{f_1^2}{f_2}(f_2^4)^2$$

and use (2.1.2) and (2.1.9), with q replaced with q^2 in the latter identity.

Likewise, for (2.3.2), write

$$\frac{1}{f_1^2 f_2^3} = f_4^2 \frac{f_2^5}{f_1^2 f_4^2}\frac{1}{(f_2^4)^2}$$

and use (2.1.1) and (2.1.10), with q replaced with q^2 in the latter identity. ☐

Theorem 2.3. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:*

$$\left\{ f_1^2 f_2^7 C(q^4),\ \frac{f_2^{13}}{f_1^2 f_4^2}C(q^4),\ \frac{1}{f_1^2 f_2^3}\frac{f_4^{22}}{f_8^8}C(q^4),\ \frac{f_1^2}{f_2^9}\frac{f_4^{24}}{f_8^8}C(q^4) \right\}. \tag{2.3.3}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.3.4}$$

Proof. From (2.3.1) and (2.3.2), elementary algebra shows that, up to sign, the first and third eta quotients at (2.3.3) have the same 4-dissections and thus have identically vanishing coefficients. The second and fourth eta quotients at (2.3.3) are the $q \to -q$ partners of the first and third, respectively. This completes the proof. ☐

Lemma 2.7. *The following 4-dissections hold:*

$$\frac{f_2}{f_1^2} = \frac{f_8^4}{f_4^{10}}\left(\frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right), \tag{2.3.5}$$

$$f_1^2 f_2^3 = \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right) = \frac{f_8^{15}}{f_4^4 f_{16}^6} - \frac{2q f_8^9}{f_4^2 f_{16}^2} - 4q^2 f_8^3 f_{16}^2 + \frac{8q^3 f_4^2 f_{16}^6}{f_8^3}, \tag{2.3.6}$$

$$\frac{f_1^6}{f_2^3} = \frac{f_8^{15}}{f_4^6 f_{16}^6} - 6q\frac{f_8^9}{f_4^4 f_{16}^2} + 12q^2\frac{f_8^3 f_{16}^2}{f_4^2} - 8q^3\frac{f_{16}^6}{f_8^3}. \tag{2.3.7}$$

Proof. The first expansion at (2.3.5) follows upon writing

$$\frac{f_2}{f_1^2} = f_4^2 \frac{f_2^5}{f_1^2 f_4^2}\frac{1}{f_2^4},$$

and using (2.1.1) and (2.1.10), with q replaced with q^2 in the latter identity. Likewise, (2.3.6) follows after writing

$$f_1^2 f_2^3 = \frac{f_1^2}{f_2} f_2^4,$$

and using (2.1.2) and (2.1.9), also with q replaced with q^2 in the latter identity.

Finally, (2.3.7) follows similarly after writing

$$\frac{f_1^6}{f_2^3} = \left(\frac{f_1^2}{f_2}\right)^3,$$

and using (2.1.2). □

Theorem 2.4. *Let $C(q^4)$ be any eta quotient whose series expansion contains only powers of q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:*

$$\left\{\frac{f_2}{f_1^2}C(q^4),\ \frac{f_1^2 f_4^2}{f_2^5}C(q^4),\ \frac{f_1^2 f_2^3 f_8^4}{f_4^{10}}C(q^4),\ \frac{f_2^9 f_8^4}{f_1^2 f_4^{12}}C(q^4),\ \frac{f_1^6 f_8^4}{f_2^3 f_4^8}C(q^4),\ \frac{f_2^{15} f_8^4}{f_1^6 f_4^{14}}C(q^4)\right\}. \tag{2.3.8}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.3.9}$$

Proof. The claim holds if $F(q)$ and $G(q)$ are any two of the first, third and fifth eta quotients in the list, by (2.3.5)-(2.3.7). The full claim follows since the second, fourth and sixth eta quotients in the list are the $q \to -q$ partners of the first, third and fifth, respectively. □

Lemma 2.8. *The following 4-dissections hold:*

$$\frac{f_2^3 f_3}{f_1} = \frac{f_4^2 f_{16}^3 f_{24}^6}{f_8^3 f_{12}^2 f_{48}^3} + q\frac{f_4 f_{16} f_{24}^3}{f_{12} f_{48}} - q^2\frac{f_8^3 f_{48}}{f_{16}} - q^3\frac{f_8^6 f_{12} f_{48}^3}{f_4 f_{16}^3 f_{24}^3}, \tag{2.3.10}$$

$$\frac{f_2 f_3}{f_1 f_6^2} = \frac{f_{16}^3 f_{24}^7}{f_4 f_8^2 f_{12}^5 f_{48}^3} + q\frac{f_8 f_{16} f_{24}^4}{f_4^2 f_{12}^4 f_{48}} + q^2\frac{f_8^4 f_{24} f_{48}}{f_4^3 f_{12}^3 f_{16}} + q^3\frac{f_8^7 f_{48}^3}{f_4^4 f_{12}^2 f_{16}^2 f_{24}^2}, \tag{2.3.11}$$

$$\frac{f_1^3 f_6^6}{f_2^3 f_3^3} = \frac{f_{16}^3 f_{24}^6}{f_8^3 f_{48}^3} - 3q\frac{f_{12} f_{16} f_{24}^3}{f_4 f_{48}} + 3q^2\frac{f_8^3 f_{12}^2 f_{48}}{f_4^2 f_{16}} - q^3\frac{f_8^6 f_{12}^3 f_{48}^3}{f_4^3 f_{16}^3 f_{24}^3}. \tag{2.3.12}$$

Proof. From (2.1.12),

$$\frac{f_2^3 f_3}{f_1} = \left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right) f_2 f_6 = \left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\left(\frac{f_4 f_{16}^2 f_{24}^4}{f_8^2 f_{12} f_{48}^2} - q^2\frac{f_8^4 f_{12} f_{48}^2}{f_4 f_{16}^2 f_{24}^2}\right),$$

where the second equality is a result of replacing q with q^2 in (2.1.7) and applying the result to the $f_2 f_6$ factor. Identity (2.3.10) follows.

Likewise,

$$\frac{f_2 f_3}{f_1 f_6^2} = \left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\frac{1}{f_2 f_6} = \left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\frac{f_8 f_{24}}{f_4^3 f_{12}^3}\left(\frac{f_4 f_{16}^2 f_{24}^4}{f_8^2 f_{12} f_{48}^2} + \frac{q^2 f_8^4 f_{12} f_{48}^2}{f_4 f_{16}^2 f_{24}^2}\right),$$

where the second equality is a consequence of replacing q by q^2 in (2.1.8) and applying the result to the $1/(f_2 f_6)$ factor. Identity (2.3.11) similarly follows.

The final identity (2.3.12) is an immediate consequence of using (2.1.11) to write

$$\frac{f_1^3 f_6^6}{f_2^3 f_3^3} = \left(\frac{f_1 f_6^2}{f_2 f_3}\right)^3 = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}}\right)^3.$$

□

Theorem 2.5. *Let $C(q^4)$ be any eta quotient whose series expansion contains only powers of q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:*
(2.3.13)
$$\left\{ \frac{f_2^3 f_3}{f_1} C(q^4), \frac{f_1 f_4 f_6^3}{f_3 f_{12}} C(q^4), \frac{f_2 f_3 f_4^3 f_{12}^3}{f_1 f_6^2 f_8 f_{24}} C(q^4), \frac{f_1 f_4^4 f_6 f_{12}^2}{f_2^2 f_3 f_8 f_{24}} C(q^4), \frac{f_1^3 f_4^2 f_6^6}{f_2^3 f_3^3 f_{12}^2} C(q^4), \frac{f_2^6 f_3^3 f_{12}}{f_1^3 f_4 f_6^3} C(q^4). \right\}.$$
Then
$$F_{(0)} = G_{(0)}. \tag{2.3.14}$$

Proof. By (2.3.10)-(2.3.12), the claim holds if $F(q)$ and $G(q)$ are any two of the first, third and fifth eta quotients in the list. The claims in the theorem hold generally since the second, fourth and sixth eta quotients in the list are the $q \to -q$ partners of the first, third and fifth, respectively. □

Lemma 2.9. *The following 4-dissections hold:*
$$\frac{f_1 f_2^2 f_6}{f_3} = \frac{f_8^2 f_{16} f_{24}}{f_{48}} - q\frac{f_8^5 f_{12} f_{48}}{f_4 f_{16} f_{24}^2} - 3q^2 \frac{f_4^2 f_{16} f_{24}^5}{f_8^2 f_{12}^2 f_{48}} + 3q^3 \frac{f_4 f_8 f_{24}^2 f_{48}}{f_{12} f_{16}}, \tag{2.3.15}$$
$$\frac{f_1 f_6^3}{f_2^4 f_3} = \frac{f_8^5 f_{12}^3 f_{16}}{f_4^9 f_{48}} - q\frac{f_8^8 f_{12}^4 f_{48}}{f_4^{10} f_{16} f_{24}^3} + 3q^2 \frac{f_8 f_{12} f_{16} f_{24}^4}{f_4^7 f_{48}} - 3q^3 \frac{f_8^4 f_{12}^2 f_{24} f_{48}}{f_4^8 f_{16}}, \tag{2.3.16}$$
$$\frac{f_1}{f_3 f_6} = \frac{f_4 f_8 f_{16} f_{24}^4}{f_{12}^7 f_{48}} - q\frac{f_8^4 f_{24} f_{48}}{f_{12}^6 f_{16}} - q^2 \frac{f_4^3 f_{16} f_{24}^8}{f_8^3 f_{12}^9 f_{48}} + q^3 \frac{f_4^2 f_{24}^5 f_{48}}{f_{12}^8 f_{16}}, \tag{2.3.17}$$
$$\frac{f_2 f_3 f_6^2}{f_1} = \frac{f_8^2 f_{12} f_{16} f_{24}}{f_4 f_{48}} + q\frac{f_8^5 f_{12}^2 f_{48}}{f_4^2 f_{16} f_{24}^2} + q^2 \frac{f_4 f_{16} f_{24}^5}{f_8^2 f_{12} f_{48}} + q^3 \frac{f_8 f_{24}^2 f_{48}}{f_{16}}. \tag{2.3.18}$$

Proof. The proofs are similar to those of the 4-dissections in Lemma 2.8. From (2.1.11),
$$\frac{f_1}{f_3} f_2^2 f_6 = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \right) \frac{f_2^3}{f_6} = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \right) \left(\frac{f_8^3}{f_{24}} - \frac{3q^2 f_4^2 f_{24}^3}{f_8 f_{12}^2} \right),$$
where the second equality is a consequence of applying (2.1.3), with q replaced with q^2, to the f_2^3/f_6 factor. Identity (2.3.15) follows.

Likewise,
$$\frac{f_1}{f_3} \frac{f_6^3}{f_2^4} = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \right) \frac{f_6}{f_2^3} = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \right) \frac{f_8^3 f_{12}^3}{f_4^9 f_{24}} \left(\frac{f_8^3}{f_{24}} + \frac{3q^2 f_4^2 f_{24}^3}{f_8 f_{12}^2} \right).$$
The second equality in the calculation above is a result of replacing q with q^2 in (2.1.4) and applying the result to the f_6/f_2^3 factor. Identity (2.3.16) follows.

Next, by similar reasoning,
$$\frac{f_1}{f_3} \frac{1}{f_6} = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \right) \frac{f_2}{f_6^3} = \left(\frac{f_{16} f_{24}^2}{f_8 f_{48}} - \frac{q f_8^2 f_{12} f_{48}}{f_4 f_{16} f_{24}} \right) \frac{f_4^3 f_{24}^3}{f_8 f_{12}^9} \left(\frac{f_8^3 f_{12}^2}{f_4^2 f_{24}} - \frac{q^2 f_{24}^3}{f_8} \right),$$
this time the second equality being a consequence of applying (2.1.6), once again with q replaced with q^2, to the f_2/f_6^3 factor. Identity (2.3.17) likewise follows.

Finally, employing (2.1.12) and replacing q by q^2 in (2.1.5) and applying the result to the f_6^3/f_2 factor below,
$$\frac{f_3}{f_1} f_2 f_6^2 = \left(\frac{f_4 f_{16} f_{24}^2}{f_{12} f_{48} f_8} + \frac{f_{48} f_8^2 q}{f_{16} f_{24}} \right) \frac{f_6^3}{f_2} = \left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + \frac{q f_8^2 f_{48}}{f_{16} f_{24}} \right) \left(\frac{f_8^3 f_{12}^2}{f_4^2 f_{24}} + \frac{q^2 f_{24}^3}{f_8} \right).$$
This proves (2.3.18). □

Theorem 2.6. *Let $C(q^4)$ be any eta quotient whose series expansion contains only powers of q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:*

$$\left\{ \frac{f_1 f_2^2 f_6}{f_3} C(q^4),\ \frac{f_2^5 f_3 f_{12}}{f_1 f_4 f_6^2} C(q^4),\ \frac{f_1 f_4^9 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3} C(q^4),\ \frac{f_3 f_4^8 f_{24}}{f_1 f_2 f_8^3 f_{12}^2} C(q^4), \frac{f_1 f_8 f_{12}^7}{f_3 f_4 f_6 f_{24}^3} C(q^4),\ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^2 f_6^4 f_{24}^3} C(q^4),\ \frac{f_2 f_3 f_4 f_6^2}{f_1 f_{12}} C(q^4),\ \frac{f_1 f_4^2 f_6^5}{f_2^2 f_3 f_{12}^2} C(q^4). \right\}. \tag{2.3.19}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.3.20}$$

Proof. By (2.3.15)-(2.3.18), the claim holds if $F(q)$ and $G(q)$ are any two of the first, third, fifth and seventh eta quotients in the list. The claims in the theorem hold generally since the second, fourth, sixth and eighth eta quotients in the list are the $q \to -q$ partners of the first, third, fifth and seventh, respectively. □

The next dissection results are not quite so straightforward, and some preliminary identities are necessary to simplify some of the components of these 4-dissections.

Lemma 2.10. *The following identities hold.*

$$\frac{f_1^3 f_6^5}{f_3 f_{12}^2} - \frac{f_2^9 f_3 f_6^2}{f_1^3 f_4^3 f_{12}} = -\frac{6q f_2^2 f_6^3 f_{12}}{f_4}, \tag{2.3.21}$$

$$\frac{f_2^3 f_3^3}{f_1 f_4^2} + \frac{f_1 f_6^9}{f_3^3 f_4 f_{12}^3} = \frac{2 f_2 f_4 f_6^2}{f_{12}}, \tag{2.3.22}$$

$$\frac{f_2^6 f_3 f_{12}}{f_1^3 f_4 f_6} - \frac{f_3^3 f_4^2}{f_1 f_2} = \frac{2q f_4 f_{12}^3}{f_2}, \tag{2.3.23}$$

$$\frac{3 f_1 f_4 f_6^7}{f_3^3} - \frac{f_1^3 f_2 f_{12}^3}{f_3} = 2 f_2 f_4^3 f_{12}^2. \tag{2.3.24}$$

Proof. These all follow directly from various parts of Lemma 2.2, upon using the 2-dissections stated in this lemma to replace f_1^3/f_3, f_3/f_1^3, f_3^3/f_1 and f_1/f_3^3 in each of the left sides above, and then simplifying. □

Lemma 2.11. *The following 4-dissections hold:*

$$\frac{f_1 f_3^3}{f_2} = \frac{f_8^9 f_{24}^2}{f_4^4 f_{16}^3 f_{48}} - q \frac{f_8^2 f_{24}^9}{f_4 f_{12}^3 f_{16} f_{48}^3} + 4q^2 \frac{q^4 f_{16} f_{48}^3}{f_8} - 4q^3 \frac{f_{12} f_{16}^3 f_{48}}{f_4 f_{24}}, \tag{2.3.25}$$

$$\frac{f_1^3 f_3}{f_6} = \frac{f_8^9 f_{24}^2}{f_4^3 f_{12} f_{16}^3 f_{48}} - 3q \frac{f_8^2 f_{24}^9}{f_{12}^4 f_{16} f_{48}^3} - 12q^2 \frac{q^4 f_4 f_{16} f_{48}^3}{f_8 f_{12}} + 4q^3 \frac{f_{16}^3 f_{48}}{f_{24}}. \tag{2.3.26}$$

Proof. By first using (2.1.2) and (2.1.5), write

$$\begin{aligned}
\frac{f_1 f_3^3}{f_2} = \frac{f_1^2}{f_2} \frac{f_3^3}{f_1} &= \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_4^3}{f_{12}} \frac{f_6^2}{f_2^2} + q \frac{f_{12}^3}{f_4} \right) \\
&= \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_{16}^2}{f_8} \right) \left(\frac{f_4^3}{f_{12}} \left(\frac{f_8^4 f_{12} f_{24}^2}{f_4^5 f_{16} f_{48}} + 2q^2 \frac{f_8 f_{12}^2 f_{16} f_{48}}{f_4^4 f_{24}} \right) + q \frac{f_{12}^3}{f_4} \right) \\
&= \frac{f_8^9 f_{24}^2}{f_4^4 f_{16}^3 f_{48}} + q \left(\frac{f_8^5 f_{12}^3}{f_4^3 f_{16}^2} - \frac{2 f_8^3 f_{16} f_{24}^2}{f_4^2 f_{48}} \right) + q^2 \left(-\frac{2 f_{12}^3 f_{16}^2}{f_4 f_8} + \frac{2 f_8^6 f_{12} f_{48}}{f_4^3 f_{16} f_{24}} \right) - 4q^3 \frac{f_{12} f_{16}^3 f_{48}}{f_4 f_{24}},
\end{aligned}$$

where the third expression above follows from the second upon using (2.1.14), with q replaced with q^2, to replace f_6^2/f_2^2. The fourth expression above follows from the third upon multiplying out and collecting powers of q in the same arithmetic progressions modulo 4. That the coefficient of q above is equal to the coefficient of q on the right side of (2.3.25) follows from multiplying the identity at (2.3.22) across by f_2^2/f_1^2, then replacing q with q^4, and finally slightly rearranging the resulting identity. Likewise, that the coefficient of q^2 above is equal to the coefficient of q^2 on the right side of (2.3.25) follows from multiplying the identity at (2.3.23) across by 2 and then replacing q with q^4.

In a similar manner, using (2.1.2), with q replaced with q^3, and employing (2.1.3), write

$$\begin{aligned}\frac{f_1^3 f_3}{f_6} = \frac{f_1^3}{f_3}\frac{f_3^2}{f_6} &= \left(\frac{f_4^3}{f_{12}} - 3q\frac{f_{12}^3}{f_4}\frac{f_2^2}{f_6^2}\right)\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_{48}^2}{f_{24}}\right)\\ &= \left(\frac{f_4^3}{f_{12}} - 3q\frac{f_{12}^3}{f_4}\left(\frac{f_4 f_8^2 f_{24}^4}{f_{12}^5 f_{16} f_{48}} - 2q^2\frac{f_4^2 f_{16} f_{24} f_{48}}{f_8 f_{12}^4}\right)\right)\left(\frac{f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_{48}^2}{f_{24}}\right)\\ = \left(\frac{f_4^3 f_{24}^5}{f_{12}^3 f_{48}^2} + 6q^4\frac{f_8^2 f_{24}^3 f_{48}}{f_{12}^2 f_{16}}\right) &- 3q\frac{f_8^2 f_{24}^9}{f_{12}^4 f_{16} f_{48}^3} - 12q^2\frac{q^4 f_4 f_{16} f_{48}^3}{f_8 f_{12}} + q^3\left(\frac{6 f_4 f_{16} f_{24}^6}{f_8 f_{12}^3 f_{48}} - \frac{2 f_4^3 f_{48}^2}{f_{12} f_{24}}\right).\end{aligned}$$

The third expression follows from the second upon using (2.1.13), with q replaced with q^2, to replace f_2^2/f_6^2. The fourth expression again follows from the third upon multiplying out and collecting powers of q in the same arithmetic progressions modulo 4. That the first term of the 4-dissection above is equal to the first term in the 4-dissection on the right side of (2.3.26) follows from multiplying the identity at (2.3.21) across by $1/f_3^2$, then replacing q with q^4 and slightly rearranging the resulting identity. Likewise, that the coefficient of q^3 in the 4-dissection above is equal to the coefficient of q^3 on the right side of (2.3.26) follows from multiplying the identity at (2.3.24) across by $2/(f_2 f_6 f_{12})$ and then replacing q with q^4. □

Theorem 2.7. *Let $C(q^4)$ be any eta quotient whose series expansion contains only powers of q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:*

$$\left\{\frac{f_1 f_3^3}{f_2}C(q^4), \frac{f_2^2 f_6^9}{f_1 f_3^3 f_4 f_{12}^3}C(q^4), \frac{f_1^3 f_3 f_{12}}{f_4 f_6}C(q^4), \frac{f_2^9 f_6^2}{f_1^3 f_3 f_4^4}C(q^4)\right\}. \tag{2.3.27}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.3.28}$$

Proof. The claim holds if $F(q)$ and $G(q)$ are the first and third eta quotients in the list, by (2.3.25) and (2.3.26), and holds generally since the second and fourth eta quotients in the list are the $q \to -q$ partners of the first and third, respectively. □

The next collection of 4-dissection results are of a slightly different character as the components of the dissections are not simple quotients of the $\prod_j f_j^{n_j}$, but instead involve more general Jacobi triple products. Further, the first collection of 4-dissection results above were discovered by examining series expansions of eta quotients in the table of such eta quotients with vanishing coefficient behaviour similar to that of f_1^4, while the next collection of 4-dissection results were derived by similarly examining eta quotients in the corresponding table for f_1^6.

We recall the notation, for a an integer and m a positive integer (see, for example, [4]),

$$\bar{J}_{a,m} := (-q^a, -q^{m-a}, q^m; q^m)_\infty. \tag{2.3.29}$$

We need the following 2-dissections of f_1 and $1/f_1$.

Lemma 2.12. *The following 2-dissections hold.*

$$f_1 = \frac{f_2}{f_4}\left(\bar{J}_{6,16} - q\bar{J}_{2,16}\right), \tag{2.3.30}$$

$$\frac{1}{f_1} = \frac{1}{f_2^2}\left(\bar{J}_{6,16} + q\bar{J}_{2,16}\right). \tag{2.3.31}$$

Proof. The identity (2.3.31) was proven by Hirschhorn [5, Lemma 1], and (2.3.30) is its $q \to -q$ partner. □

Before we get to the next lemma, we make a general comment. In several of the previous lemmas, and likewise in the upcoming Lemma 2.13, dissections of many eta quotients are derived by "multiplying together" (in a certain sense) existing dissections, after rewriting the eta quotient as a product of eta quotients which already have known dissections. We state a simple example of a family of four eta quotients, containing a free integer parameter, derived in this way such that all four have identically vanishing coefficients.

Example 1. *Let m be a positive integer and let $F(q)$ and $G(q)$ be any two eta quotients in the following list:*

$$\left\{\frac{f_1^2 f_{2m}}{f_2}, \frac{f_2^5 f_{2m}}{f_1^2 f_4^2}, \frac{f_1^2 f_{4m}^3}{f_2 f_{2m} f_{8m}}, \frac{f_2^5 f_{4m}^3}{f_1^2 f_4^2 f_{2m} f_{8m}}\right\}. \tag{2.3.32}$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. This follows directly after using (2.1.1), (2.1.2), and replacing q q^{2m} in (2.3.30) and (2.3.31). It follows that all four eta quotients have similar 4-dissections. □

Note that all the eta quotients at (2.3.32) are lacunary for every positive integer m, following an observation of Ono and Robins [14, page 1023] on the product of two *superlacunary* eta quotients.

The 2-dissections at (2.3.30) and (2.3.31) above are central in the proof of the following 4-dissections.

Lemma 2.13. *The following 4-dissections hold.*

$$\frac{f_1^2}{f_2^2} = \frac{1}{f_4^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{12,32} + q^2\bar{J}_{4,32}\right), \tag{2.3.33}$$

$$f_1^2 = \frac{f_4}{f_8}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{12,32} - q^2\bar{J}_{4,32}\right), \tag{2.3.34}$$

$$\frac{f_1^2}{f_2^4} = \frac{f_8^3}{f_4^{11}}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)\left(\bar{J}_{12,32} - q^2\bar{J}_{4,32}\right), \tag{2.3.35}$$

$$f_1^2 f_2^2 = \frac{1}{f_4^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)\left(\bar{J}_{12,32} + q^2\bar{J}_{4,32}\right), \tag{2.3.36}$$

$$\frac{f_3}{f_1} = \frac{f_{12}}{f_4^4}\left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} + q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} + q^2\bar{J}_{4,32}\right), \tag{2.3.37}$$

$$\frac{f_1 f_2 f_6}{f_3} = \frac{f_4 f_{24}}{f_8^2 f_{12}}\left(\frac{f_4 f_{16} f_{24}^2}{f_8 f_{12} f_{48}} - q\frac{f_8^2 f_{48}}{f_{16} f_{24}}\right)\left(\frac{f_8 f_{32} f_{48}^2}{f_{16} f_{24} f_{96}} - q^2\frac{f_{16}^2 f_{96}}{f_{32} f_{48}}\right)\left(\bar{J}_{12,32} - q^2\bar{J}_{4,32}\right), \tag{2.3.38}$$

$$\frac{f_1}{f_3}=\frac{f_4}{f_{12}^4}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\frac{f_{32}f_{48}^2}{f_{16}f_{96}}-q^2\frac{f_{16}^2f_{24}f_{96}}{f_8f_{32}f_{48}}\right)\left(\bar{J}_{36,96}+q^6\bar{J}_{12,96}\right), \tag{2.3.39}$$

$$\frac{f_2f_3f_6}{f_1}=\frac{f_8f_{12}}{f_4f_{24}^2}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}+q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\frac{f_{32}f_{48}^2}{f_{16}f_{96}}+q^2\frac{f_{16}^2f_{24}f_{96}}{f_8f_{32}f_{48}}\right)\left(\bar{J}_{36,96}-q^6\bar{J}_{12,96}\right), \tag{2.3.40}$$

$$\frac{f_1^2}{f_6^2}=\frac{f_4}{f_{12}^4}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{32}f_{48}^2}{f_{16}f_{96}}-q^2\frac{f_{16}^2f_{24}f_{96}}{f_8f_{32}f_{48}}\right)\left(\bar{J}_{36,96}+q^6\bar{J}_{12,96}\right), \tag{2.3.41}$$

$$\frac{f_1^2f_6^2}{f_2^2}=\frac{f_8f_{12}^2}{f_4^2f_{24}^2}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{32}f_{48}^2}{f_{96}f_{16}}+q^2\frac{f_{16}^2f_{24}f_{96}}{f_8f_{32}f_{48}}\right)\left(\bar{J}_{36,96}-q^6\bar{J}_{12,96}\right), \tag{2.3.42}$$

$$\frac{f_3}{f_1f_2^3f_6}=\frac{f_8^4}{f_4^{14}}\left(\frac{f_8^{10}}{f_4^2f_{16}^4}+\frac{4f_4^2f_{16}^4q^2}{f_8^2}\right)\left(\frac{f_4f_{16}f_{24}^2}{f_{12}f_{48}f_8}+\frac{f_{48}f_8^2q}{f_{16}f_{24}}\right)\left(J_{12,32}+q^2J_{4,32}\right), \tag{2.3.43}$$

$$\frac{f_2^7f_3}{f_1f_6}=\frac{f_4}{f_8}\left(\frac{f_8^{10}}{f_4^2f_{16}^4}-\frac{4f_4^2f_{16}^4q^2}{f_8^2}\right)\left(\frac{f_4f_{16}f_{24}^2}{f_{12}f_{48}f_8}+\frac{f_{48}f_8^2q}{f_{16}f_{24}}\right)\left(J_{12,32}-q^2J_{4,32}\right), \tag{2.3.44}$$

$$\frac{f_2^5f_3^2}{f_6}=\frac{f_4}{f_8}\left(\frac{f_8^{10}}{f_4^2f_{16}^4}-4q^2\frac{f_4^2f_{16}^4}{f_8^2}\right)\left(\frac{f_{24}^5}{f_{12}^2f_{48}^2}-2q^3\frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32}-q^2J_{4,32}\right), \tag{2.3.45}$$

$$\frac{f_6^5}{f_2^5f_3^2}=\frac{f_8^4f_{12}^2}{f_4^{14}}\left(\frac{f_8^{10}}{f_4^2f_{16}^4}+4q^2\frac{f_4^2f_{16}^4}{f_8^2}\right)\left(\frac{f_{24}^5}{f_{12}^2f_{48}^2}+2q^3\frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32}+q^2J_{4,32}\right), \tag{2.3.46}$$

$$\frac{f_2f_3^2}{f_6}=\frac{f_4}{f_8}\left(\frac{f_{24}^5}{f_{12}^2f_{48}^2}-2q^3\frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32}-q^2J_{4,32}\right), \tag{2.3.47}$$

$$\frac{f_6^5}{f_2f_3^2}=\frac{f_{12}^2}{f_4^2}\left(\frac{f_{24}^5}{f_{12}^2f_{48}^2}+2q^3\frac{f_{48}^2}{f_{24}}\right)\left(J_{12,32}+q^2J_{4,32}\right), \tag{2.3.48}$$

$$\frac{f_1f_6^2}{f_2^4f_3}=\frac{1}{f_4^6}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\bar{J}_{12,32}+q^2\bar{J}_{4,32}\right)^3, \tag{2.3.49}$$

$$\frac{f_1f_2^2f_6^2}{f_3}=\frac{f_4^3}{f_8^3}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\bar{J}_{12,32}-q^2\bar{J}_{4,32}\right)^3, \tag{2.3.50}$$

$$\frac{f_1^2f_6}{f_2^3}=\frac{f_{12}}{f_4^4}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8f_{32}f_{48}^2}{f_{16}f_{24}f_{96}}+q^2\frac{f_{16}^2f_{96}}{f_{32}f_{48}}\right)\left(\bar{J}_{12,32}+q^2\bar{J}_{4,32}\right), \tag{2.3.51}$$

$$\frac{f_1^2f_2}{f_6}=\frac{f_4^2f_{24}}{f_8^2f_{12}^2}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8f_{32}f_{48}^2}{f_{16}f_{24}f_{96}}-q^2\frac{f_{16}^2f_{96}}{f_{32}f_{48}}\right)\left(\bar{J}_{12,32}-q^2\bar{J}_{4,32}\right), \tag{2.3.52}$$

$$\frac{f_1f_6^2}{f_3}=\frac{f_4}{f_8}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\bar{J}_{12,32}-q^2\bar{J}_{4,32}\right), \tag{2.3.53}$$

$$\frac{f_1f_6^2}{f_2^2f_3}=\frac{1}{f_4^2}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\bar{J}_{12,32}+q^2\bar{J}_{4,32}\right), \tag{2.3.54}$$

(2.3.55)

$$\frac{f_2^2f_3}{f_1f_6^6}=\frac{f_4^2f_{24}^6}{f_8^2f_{12}^{17}}\left(\frac{f_8^3f_{12}^2}{f_4^2f_{24}}-q^2\frac{f_{24}^3}{f_8}\right)^2\left(\frac{f_4f_{16}f_{24}^2}{f_{12}f_{48}f_8}+q\frac{f_{48}f_8^2}{f_{16}f_{24}}\right)\left(\frac{f_8f_{32}f_{48}^2}{f_{24}f_{96}f_{16}}+q^2\frac{f_{96}f_{16}^2}{f_{32}f_{48}}\right)\left(\bar{J}_{12,32}+q^2\bar{J}_{4,32}\right),$$

(2.3.56)

$$\frac{f_1f_6^7}{f_2f_3}=\frac{f_4f_{24}}{f_8^2f_{12}}\left(\frac{f_{12}^2f_8^3}{f_4^2f_{24}}+q^2\frac{f_{24}^3}{f_8}\right)^2\left(\frac{f_4f_{16}f_{24}^2}{f_8f_{12}f_{48}}-q\frac{f_8^2f_{48}}{f_{16}f_{24}}\right)\left(\frac{f_8f_{32}f_{48}^2}{f_{16}f_{24}f_{96}}-q^2\frac{f_{16}^2f_{96}}{f_{32}f_{48}}\right)\left(\bar{J}_{12,32}-q^2\bar{J}_{4,32}\right),$$

$$\frac{f_1^2}{f_2f_6^5}=\frac{f_{24}^4}{f_{12}^{14}}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{24}^{10}}{f_{12}^2f_{48}^4}+4q^6\frac{f_{12}^2f_{48}^4}{f_{24}^2}\right)\left(\bar{J}_{36,96}+q^6\bar{J}_{12,96}\right), \tag{2.3.57}$$

$$\frac{f_1^2f_6^5}{f_2}=\frac{f_{12}}{f_{24}}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{24}^{10}}{f_{12}^2f_{48}^4}-4q^6\frac{f_{12}^2f_{48}^4}{f_{24}^2}\right)\left(\bar{J}_{36,96}-q^6\bar{J}_{12,96}\right), \tag{2.3.58}$$

$$\frac{f_1^2}{f_2f_6}=\frac{1}{f_{12}^2}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{36,96}+q^6\bar{J}_{12,96}\right), \tag{2.3.59}$$

$$\frac{f_1^2f_6}{f_2}=\frac{f_{12}}{f_{24}}\left(\frac{f_8^5}{f_4^2f_{16}^2}-2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{36,96}-q^6\bar{J}_{12,96}\right), \tag{2.3.60}$$

$$\frac{f_1f_6}{f_2f_3}=\frac{1}{f_{12}^2}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\bar{J}_{36,96}+q^6\bar{J}_{12,96}\right), \tag{2.3.61}$$

$$\frac{f_1f_6^3}{f_2f_3}=\frac{f_{12}}{f_{24}}\left(\frac{f_{16}f_{24}^2}{f_8f_{48}}-q\frac{f_8^2f_{12}f_{48}}{f_4f_{16}f_{24}}\right)\left(\bar{J}_{36,96}-q^6\bar{J}_{12,96}\right). \tag{2.3.62}$$

Proof. The expansion (2.3.33) follows upon writing $f_1^2/f_2^2=(f_1^2/f_2)(1/f_2)$ and then employing (2.1.2) and (2.3.31), with q replaced with q^2 in the latter identity. The expansion (2.3.34) follows similarly after writing $f_1^2=(f_1^2/f_2)f_2$ and then employing (2.1.2) and applying (2.3.30), again with q replaced with q^2.

The dissection (2.3.35) is a consequence of writing f_1^2/f_2^4 as $f_1^2(1/f_2^4)$ and using (2.3.34) and (2.1.10), with q replaced with q^2. The dissection (2.3.36) is similarly a consequence of writing $f_1^2f_2^2$ as $(f_1^2/f_2^2)f_2^4$ and using (2.3.33) and (2.1.9), also with q replaced with q^2.

Use (2.1.11) and (2.1.12) to write

$$\frac{f_3}{f_1}=\left(\frac{f_4f_{16}f_{24}^2}{f_8f_{12}f_{48}}+q\frac{f_8^2f_{48}}{f_{16}f_{24}}\right)\frac{f_6}{f_2}\frac{1}{f_2},\qquad\frac{f_1f_2f_6}{f_3}=\left(\frac{f_4f_{16}f_{24}^2}{f_8f_{12}f_{48}}-q\frac{f_8^2f_{48}}{f_{16}f_{24}}\right)\frac{f_2}{f_6}f_2.$$

Then (2.3.37) follows upon replacing q by q^2 in (2.1.12) and replacing q by q^2 in (2.3.31) and applying the results to the second and third factors on the right side of the first equation above. In a similar fashion, (2.3.38) follows after replacing q by q^2 in (2.1.11) and a second time and applying the results to the second and third factors on the right side of the second equation above. Some further simple algebraic manipulation is needed to make the similarity of the right sides of (2.3.37) and (2.3.38) more apparent. The proofs of (2.3.39) and (2.3.40) are very similar and so are omitted.

The dissections at (2.3.41) and (2.3.42) follow upon writing

$$\frac{f_1^2}{f_6^2}=\frac{f_1^2}{f_2}\frac{f_2}{f_6}\frac{1}{f_6},\qquad\frac{f_1^2f_6^2}{f_2^2}=\frac{f_1^2}{f_2}\frac{f_6}{f_2}f_6,$$

and then using (2.1.2) and either (2.1.11), with q replaced with q^2, and (2.3.31), with q replaced with q^6; or (2.1.12), with q replaced with q^2, and (2.3.30), with q replaced with q^6.

The proofs of the remaining dissections are similar. We sketch the idea and then subsequently confine ourselves to listing the identities involved when the method is applied. In the case of each 4-dissection, a specified identity or identities is used to express the left side, say $L(q)$, in the form

$L(q) = A_1(q^4)A_2(q^2)$, and then a known 2-dissection for $A_2(q)$, with q replaced with q^2, is used to give a 4-dissection of $A_2(q^2)$. This procedure produces the stated 4-dissection of $L(q)$.

Identity (2.3.43) uses (2.1.12), (2.1.10) and (2.3.31), with q replaced with q^2 in the latter two identities. Identity (2.3.44) similarly requires (2.1.12), (2.1.9) and (2.3.30).

For (2.3.45), write $f_2^5 f_3^2/f_6 = (f_3^2/f_6)(f_2^4)(f_2)$ and use (2.1.2), with q replaced with q^3, and employ (2.1.9) and (2.3.30), both with q replaced with q^2. Identity (2.3.46) follows similarly upon writing $f_6^5/(f_2^5 f_3^2) = (f_6^5/(f_3^2 f_{12}^2))\,(1/f_2^4)(1/f_2)f_{12}^2$ and using (2.1.1), (2.1.10) and (2.3.31).

The proofs of the identities (2.3.47) and (2.3.48) are similar to those of (2.3.45) and (2.3.46), except simpler, writing, respectively, $f_2 f_3^2/f_6 = (f_3^2/f_6)f_2$ and $f_6^5/(f_2 f_3^2) = (f_6^5/(f_3^2 f_{12}^2))\,(1/f_2)f_{12}^2$.

The dissection (2.3.49) follows as consequence of employing (2.1.11) and (2.3.31), while (2.3.50) is derived similarly from using (2.1.11) in combination with (2.3.30).

Identity (2.3.51) follows upon writing $f_1^2 f_6/f_2^3$ as $(f_1^2/f_2)(f_6/f_2)(1/f_2)$ and using (2.1.2), (2.1.12) and (2.3.31). Identity (2.3.52) follows upon writing $f_1^2 f_2/f_6$ as $(f_1^2/f_2)(f_2/f_6)(f_2)$ and using (2.1.2), (2.1.11) and (2.3.30).

As with (2.3.49) and (2.3.50), the claims in (2.3.53) and (2.3.54) follow from using (2.1.11) in combination with either (2.3.30) or (2.3.31).

For (2.3.55) and (2.3.56), write

$$\frac{f_2^2 f_3}{f_1 f_6^6} = \left(\frac{f_2}{f_6^3}\right)^2 \left(\frac{f_3}{f_1}\frac{f_2^2}{f_6}\right)\left(\frac{f_6}{f_2}\right)\left(\frac{1}{f_2}\right), \qquad \frac{f_1 f_6^7}{f_2 f_3} = \left(\frac{f_6^3}{f_2}\right)^2 \left(\frac{f_1}{f_3}\frac{f_6^2}{f_2}\right)\left(\frac{f_2}{f_6}\right)(f_2),$$

and then use (2.1.11), (2.1.12), (2.3.30) and (2.3.31).

To get (2.3.57) and (2.3.58), use (2.1.2), (2.1.9), (2.1.10), (2.3.30) and (2.3.31) (with q replaced with q^6 in some instances).

The proofs of (2.3.59) and (2.3.60) use (2.1.2) in combination with either (2.3.30) or (2.3.31), also with q replaced with q^6 in the latter identity.

The dissections (2.3.61) and (2.3.62) follow from using (2.1.11) with either (2.3.30) or (2.3.31), again with q replaced with q^6 in the latter identity.

□

Theorem 2.8. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:*

$$\left\{ f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8} C\left(q^4\right), \frac{f_2^4 f_4}{f_1^2 f_8} C\left(q^4\right) \right\}, \tag{2.3.63}$$

$$\left\{ f_1^2 f_2^2 C\left(q^4\right), \frac{f_2^8}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^9}{f_2^4 f_8^3} C\left(q^4\right), \frac{f_2^2 f_4^7}{f_1^2 f_8^3} C\left(q^4\right) \right\}, \tag{2.3.64}$$

$$\left\{ \frac{f_3}{f_1} C\left(q^4\right), \frac{f_1 f_4 f_6^3}{f_2^3 f_3 f_{12}} C\left(q^4\right), \frac{f_1 f_2 f_6 f_8^2 f_{12}^2}{f_3 f_4^5 f_{24}} C\left(q^4\right), \frac{f_2^4 f_3 f_8^2 f_{12}^3}{f_1 f_4^6 f_6^2 f_{24}} C\left(q^4\right). \right\}, \tag{2.3.65}$$

$$\left\{ \frac{f_1}{f_3} C\left(q^4\right), \frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6^3} C\left(q^4\right), \frac{f_2 f_3 f_4^2 f_6 f_{24}^2}{f_1 f_8 f_{12}^5} C\left(q^4\right), \frac{f_1 f_4^3 f_6^4 f_{24}^2}{f_2^2 f_3 f_8 f_{12}^6} C\left(q^4\right) \right\}, \tag{2.3.66}$$

$$\left\{ \frac{f_1^2}{f_6^2} C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2 f_6^2} C\left(q^4\right), \frac{f_1^2 f_4^3 f_6^2 f_{24}^2}{f_2^2 f_8 f_{12}^6} C\left(q^4\right), \frac{f_2^4 f_4 f_6^2 f_{24}^2}{f_1^2 f_8 f_{12}^6} C\left(q^4\right) \right\}, \tag{2.3.67}$$

$$\left\{ \frac{f_3}{f_1 f_2^3 f_6} C\left(q^4\right), \frac{f_1 f_4 f_6^2}{f_2^6 f_3 f_{12}} C\left(q^4\right), \frac{f_2^7 f_3 f_8^5}{f_1 f_4^{15} f_6} C\left(q^4\right), \frac{f_1 f_2^4 f_6^2 f_8^5}{f_3 f_4^{14} f_{12}} C\left(q^4\right) \right\}, \tag{2.3.68}$$

$$\left\{ \frac{f_2^5 f_3^2}{f_6} C\left(q^4\right), \frac{f_2^5 f_6^5}{f_3^2 f_{12}^2} C\left(q^4\right), \frac{f_4^{15} f_6^5}{f_2^5 f_3^2 f_8^5 f_{12}^2} C\left(q^4\right), \frac{f_3^2 f_4^{15}}{f_2^5 f_6 f_8^5} C\left(q^4\right) \right\}, \tag{2.3.69}$$

$$\left\{\frac{f_2f_3^2}{f_6}C\left(q^4\right),\frac{f_2f_6^5}{f_3^2f_{12}^2}C\left(q^4\right),\frac{f_4^3f_6^5}{f_2f_3^2f_8f_{12}^2}C\left(q^4\right),\frac{f_3^2f_4^3}{f_2f_6f_8}C\left(q^4\right)\right\}, \tag{2.3.70}$$

$$\left\{\frac{f_1f_2^2f_6^2}{f_3}C\left(q^4\right),\frac{f_2^5f_3f_{12}}{f_1f_4f_6}C\left(q^4\right),\frac{f_1f_4^9f_6^2}{f_2^4f_3f_8^3}C\left(q^4\right),\frac{f_3f_4^8f_{12}}{f_1f_2f_6f_8^3}C\left(q^4\right)\right\}, \tag{2.3.71}$$

$$\left\{\frac{f_1^2f_6}{f_2^3}C\left(q^4\right),\frac{f_2^3f_6}{f_1^2f_4^2}C\left(q^4\right),\frac{f_1^2f_2f_8^2f_{12}^3}{f_4^6f_6f_{24}}C\left(q^4\right),\frac{f_2^7f_8^2f_{12}^3}{f_1^2f_4^8f_6f_{24}}C\left(q^4\right)\right\}, \tag{2.3.72}$$

$$\left\{\frac{f_1f_6^2}{f_3}C\left(q^4\right),\frac{f_2^3f_3f_{12}}{f_1f_4f_6}C\left(q^4\right),\frac{f_1f_4^3f_6^2}{f_2^2f_3f_8}C\left(q^4\right),\frac{f_2f_3f_4^2f_{12}}{f_1f_6f_8}C\left(q^4\right)\right\}, \tag{2.3.73}$$

$$\left\{\frac{f_1f_6^7}{f_2f_3}C\left(q^4\right),\frac{f_2^2f_3f_6^4f_{12}}{f_1f_4}C\left(q^4\right),\frac{f_2^2f_3f_{12}^{16}}{f_1f_4f_6^6f_{24}^5}C\left(q^4\right),\frac{f_1f_{12}^{15}}{f_2f_3f_6^3f_{24}^5}C\left(q^4\right)\right\}, \tag{2.3.74}$$

$$\left\{\frac{f_1^2}{f_2f_6^5}C\left(q^4\right),\frac{f_2^5}{f_1^2f_4^2f_6^5}C\left(q^4\right),\frac{f_1^2f_6^5f_{24}^5}{f_2f_{12}^{15}}C\left(q^4\right),\frac{f_2^5f_6^5f_{24}^5}{f_1^2f_4^2f_{12}^{15}}C\left(q^4\right)\right\}, \tag{2.3.75}$$

$$\left\{\frac{f_1^2}{f_2f_6}C\left(q^4\right),\frac{f_2^5}{f_1^2f_4^2f_6}C\left(q^4\right),\frac{f_1^2f_6f_{24}}{f_2f_{12}^3}C\left(q^4\right),\frac{f_2^5f_6f_{24}}{f_1^2f_4^2f_{12}^3}C\left(q^4\right)\right\}, \tag{2.3.76}$$

$$\left\{\frac{f_1f_6}{f_2f_3}C\left(q^4\right),\frac{f_2^2f_3f_{12}}{f_1f_4f_6^2}C\left(q^4\right),\frac{f_1f_6^3f_{24}}{f_2f_3f_{12}^3}C\left(q^4\right),\frac{f_2^2f_3f_{24}}{f_1f_4f_{12}^2}C\left(q^4\right)\right\}. \tag{2.3.77}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.3.78}$$

Proof. We will prove the statement for $F(q)$ and $G(q)$ equal to the first and third eta quotients in each list. The second and fourth eta quotients in each list are the $q \to -q$ partners of the first- and third eta quotient respectively.

Taking the previous statement into account, the proofs for the identities listed in (2.3.63) to (2.3.77) follow directly from, respectively, the pairs of dissections ((2.3.33), (2.3.34)) to ((2.3.61), (2.3.62)) in Lemma 2.13. □

Remark: If m is an odd integer, and q is replaced with q^m in any of the lists in Theorem 2.8 (but leaving the $C(q^4)$ factor as is), then the result still holds. For example, making this change with $m = 3$ in the list (2.3.63) leads to the following result:

If $C(q^4)$ is any eta quotient with a power series expansion in q^4, and $F(q)$ and $G(q)$ are any pair of eta quotients in the list

$$\left\{f_3^2\mathrm{C}\left(q^4\right),\frac{f_6^6}{f_3^2f_{12}^2}\mathrm{C}\left(q^4\right),\frac{f_3^2f_{12}^3}{f_6^2f_{24}}\mathrm{C}\left(q^4\right),\frac{f_6^4f_{12}}{f_3^2f_{24}}\mathrm{C}\left(q^4\right)\right\}, \tag{2.3.79}$$

then

$$F_{(0)} = G_{(0)}.$$

This kind of result is necessary to prove identically vanishing coefficient results for some collections of eta quotients.

The first pair of dissections in the next lemma were discovered while looking at eta quotients with vanishing coefficient behaviour similar to f_1^2 ([12, Lemma 4.4]). The second pair of dissection arose while examining eta quotients with vanishing coefficient behaviour similar to f_1^{10}. The first pair of dissections permits a second proof of (2.3.63).

Lemma 2.14. *Define*

$$A_0 = A_0(q) := (q^4;q^8)_\infty(-q^{12},-q^{20},q^{32};q^{32})_\infty(-q^4,-q^4,q^8;q^8)_\infty,$$
$$A_1 = A_1(q) := -(q^4;q^8)_\infty(-q^{12},-q^{20},q^{32};q^{32})_\infty(-q^8,-1,q^8;q^8)_\infty,$$
$$A_2 = A_2(q) := -(q^4;q^8)_\infty(-q^4,-q^{28},q^{32};q^{32})_\infty(-q^4,-q^4,q^8;q^8)_\infty,$$
$$A_3 = A_3(q) := (q^4;q^8)_\infty(-q^4,-q^{28},q^{32};q^{32})_\infty(-q^8,-1,q^8;q^8)_\infty. \tag{2.3.80}$$

Then

$$f_1^2 = A_0 + qA_1 + q^2A_2 + q^3A_3, \tag{2.3.81}$$

$$\frac{f_2^4 f_4}{f_1^2 f_8} = A_0 - qA_1 - q^2A_2 + q^3A_3, \tag{2.3.82}$$

$$f_1^2 f_2^4 = \left(A_0 + qA_1 + q^2A_2 + q^3A_3\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right), \tag{2.3.83}$$

$$\frac{f_4^{13}}{f_1^2 f_8^5} = \left(A_0 - qA_1 - q^2A_2 + q^3A_3\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right). \tag{2.3.84}$$

Proof. The expansions (2.3.81) and (2.3.82) were proven in [12, Lemma 4.4]. Upon multiplying (2.3.81) by f_2^4 and using (2.1.9), with q replaced with q^2, one gets (2.3.83), and (2.3.84) follows similarly upon multiplying (2.3.82) by $f_4^{12}/(f_8^4 f_2^4)$ and using (2.1.10) (with q replaced with q^2). □

Theorem 2.9. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:*

$$\left\{\frac{f_2^4 f_4}{f_1^2 f_8}C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8}C\left(q^4\right), f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2}C\left(q^4\right)\right\}, \tag{2.3.85}$$

$$\left\{\frac{f_4^{13}}{f_1^2 f_8^5}C\left(q^4\right), \frac{f_1^2 f_4^{15}}{f_2^6 f_8^5}C\left(q^4\right), f_1^2 f_2^4 C\left(q^4\right), \frac{f_2^{10}}{f_1^2 f_4^2}C\left(q^4\right)\right\}. \tag{2.3.86}$$

Then

$$F_{(0)} = G_{(0)}. \tag{2.3.87}$$

Proof. The proof for $F(q)$ equal to the first eta quotient in each list and $G(q)$ equal to the third follows in each case from Lemma 2.14. The full statement of the theorem holds in each case, since the second and fourth eta quotients in each case are, respectively, the $q \to -q$ partners of the first and third eta quotients. □

2.4. **General Inclusion Results.** Some strict inclusion results follow quite trivially from 2-dissections. If

$$A(q) = a_0 A_0(q^2) + a_1 q A_1(q^2) \tag{2.4.1}$$

for non-zero functions $A_0(q^2)$ and $A_1(q^2)$, and $F(q) = A(q)C(q^2)$ and $G(q) = A_0(q^2)C(q^2)$, then $F_{(0)} \subsetneq G_{(0)}$ (assuming $a_1 \neq 0$).

The results in the next two lemmas are not quite so trivial.

Lemma 2.15. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4, and define $F(q)$ and $G(q)$ by*

$$F(q) = f_1^3 f_2^3 C(q^4), \tag{2.4.2}$$
$$G(q) = \frac{f_1^2 f_8^{10}}{f_2 f_4^2 f_{16}^4}C(q^4).$$

Then

$$F_{(0)} \subsetneq G_{(0)}.$$

Proof. From (2.1.2),

$$\frac{f_1^2 f_8^{10}}{f_2 f_4^2 f_{16}^4} = \frac{f_8^{15}}{f_4^4 f_{16}^6} - 2q\frac{f_8^9}{f_4^2 f_{16}^2},$$

and the claim follows upon comparison with (2.3.6). ☐

Lemma 2.16. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4, and define $F(q)$ and $G(q)$ by*

$$F(q) = \frac{f_1}{f_3 f_6} C(q^4), \tag{2.4.3}$$

$$G(q) = \frac{f_1 f_4 f_6^2 f_8^2 f_{24}^2}{f_2 f_3 f_{12}^7} C(q^4).$$

Then

$$F_{(0)} \subsetneq G_{(0)}.$$

Proof. From (2.1.11),

$$\frac{f_1 f_4 f_6^2 f_8^2 f_{24}^2}{f_2 f_3 f_{12}^7} = \frac{f_4 f_8 f_{16} f_{24}^4}{f_{12}^7 f_{48}} - q\frac{f_8^4 f_{24} f_{48}}{f_{12}^6 f_{16}},$$

and the claim follows upon comparison with (2.3.17). ☐

3. Applications to Vanishing Coefficients in the Series Expansion of Lacunary Eta Quotients

We now apply the general results on m-dissections and identically vanishing coefficients in the previous section to prove some of the conjectures from [12] on lacunary eta quotients. We remark that in one sense there is nothing to be gained by specializing the $C(q^m)$ functions in the various theorems in the previous section (for example, $C(q^4)$ in any of the lists in Theorem 2.8). However, our purpose is to prove some of the vanishing coefficient relationships between collections of *lacunary* eta quotients which were found experimentally and described in [12].

3.1. Eta quotients with vanishing coefficient behaviour similar to f_1^4.

The results in the following theorems prove some of the results found experimentally and described in Table 1 and Figure 1.

Theorem 3.1. *Let $F(q)$ and $G(q)$ be any two eta quotients from one of the following lists.*

$$\left\{\frac{f_2^6 f_3 f_6^4}{f_1^3 f_4^2 f_{12}^2}, \frac{f_1^3 f_4 f_6^7}{f_2^3 f_3 f_{12}^3}, \frac{f_3^3 f_4 f_6^5}{f_1 f_2 f_{12}^3}, \frac{f_1 f_4^2 f_6^{14}}{f_2^4 f_3^3 f_{12}^6}\right\}, \tag{3.1.1}$$

$$\left\{\frac{f_2^7 f_3^3 f_{12}}{f_1 f_4^3 f_6^3}, \frac{f_1 f_2^4 f_6^6}{f_3^3 f_4^2 f_{12}^2}, \frac{f_1^3 f_2^5 f_{12}}{f_3 f_4^3 f_6}, \frac{f_2^{14} f_3 f_{12}^2}{f_1^3 f_4^6 f_6^4}\right\}, \tag{3.1.2}$$

$$\left\{\frac{f_1 f_4^7 f_6^9}{f_2^5 f_3^3 f_{12}^5}, \frac{f_3^3 f_4^6}{f_1 f_2^2 f_{12}^2}, \frac{f_1^3 f_4^6 f_6^2}{f_2^4 f_3 f_{12}^2}, \frac{f_2^5 f_3 f_4^3}{f_1^3 f_6 f_{12}}\right\}, \tag{3.1.3}$$

$$\left\{\frac{f_1 f_4 f_6^8}{f_2^2 f_3^3 f_{12}^3}, \frac{f_2 f_3^3}{f_1 f_6}, \frac{f_1^3 f_6}{f_2 f_3}, \frac{f_2^8 f_3 f_{12}}{f_1^3 f_4^3 f_6^2}\right\}, \tag{3.1.4}$$

$$\left\{\frac{f_4^{19}}{f_1^2 f_2^3 f_8^8}, \frac{f_1^2 f_4^{21}}{f_2^9 f_8^8}, \frac{f_1^2 f_2^7}{f_4^3}, \frac{f_2^{13}}{f_1^2 f_4^5}\right\}, \tag{3.1.5}$$

$$\left\{\frac{f_2 f_4^9}{f_1^2 f_8^4}, \frac{f_1^2 f_4^{11}}{f_2^5 f_8^4}, \frac{f_2^9}{f_1^2 f_4^3}, \frac{f_1^2 f_2^3}{f_4}, \frac{f_1^6 f_4}{f_2^3}, \frac{f_2^{15}}{f_1^6 f_4^5}\right\}, \tag{3.1.6}$$

$$\left\{\frac{f_2^9 f_6^2 f_8^3}{f_1^3 f_3 f_4^5 f_{24}}, \frac{f_1^3 f_3 f_8^3 f_{12}}{f_4^2 f_6 f_{24}}, \frac{f_2^2 f_6^9 f_8^3}{f_1 f_3^3 f_4^2 f_{12}^3 f_{24}}, \frac{f_1 f_3^3 f_8^3}{f_2 f_4 f_{24}}\right\}, \tag{3.1.7}$$

$$\left\{\frac{f_2 f_3 f_4^2 f_{12}^3}{f_1 f_6^2 f_8 f_{24}}, \frac{f_1 f_4^3 f_6 f_{12}^2}{f_2^2 f_3 f_8 f_{24}}, \frac{f_1 f_6^3}{f_3 f_{12}}, \frac{f_2^3 f_3}{f_1 f_4}, \frac{f_1^3 f_4 f_6^6}{f_2^3 f_3^3 f_{12}^2}, \frac{f_2^6 f_3^3 f_{12}}{f_1^3 f_4^2 f_6^3}\right\}, \tag{3.1.8}$$

$$\left\{\frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}}\right\}. \tag{3.1.9}$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. These results for (3.1.1) - (3.1.4), respectively, follow from (2.1.20) with $C(q^2)$ equalling, respectively,

$$\frac{f_2^6 f_6^4}{f_4^2 f_{12}^2}, \qquad \frac{f_2^{14} f_{12}^2}{f_4^6 f_6^4}, \qquad \frac{f_2^5 f_4^3}{f_6 f_{12}}, \qquad \frac{f_2^8 f_{12}}{f_4^3 f_6^2}.$$

The claim for (3.1.5) follows from (2.3.3) with $C(q^4) = 1/f_4^3$. Similarly, the result for (3.1.6) follows from (2.3.8) with $C(q^4) = f_4^9/f_8^4$; that for (3.1.7) follows from (2.3.27) with $C(q^4) = f_8^3/(f_4 f_{24})$, that for (3.1.8) from (2.3.13) with $C(q^4) = 1/f_4$; and that for (3.1.9) is derived from (2.3.19), also with $C(q^4) = 1/f_4$. □

Remark: The eta quotients in collections (3.1.1), (3.1.2), (3.1.5) and (3.1.6) are all in group I of Table 1 and Figure 1, while collections (3.1.3), (3.1.4), (3.1.7), (3.1.8) and (3.1.9) comprise, respectively, groups VI, IX, II, VII and VIII (so that our theorem has completed the task of showing identical vanishing of coefficients within all of these latter groups). We next prove some inclusion results.

Theorem 3.2. *Consider any one of the following pairs of collections of eta quotients* (3.1.10) - (3.1.19)*:*

$$\left\{f_1^4, \frac{f_2^{12}}{f_1^4 f_4^4}\right\}, \qquad \left\{\frac{f_2^2 f_4^4}{f_8^2}, \frac{f_4^{10}}{f_2^2 f_8^4}\right\}, \tag{3.1.10}$$

$$\left\{\frac{f_2^6 f_3 f_6^4}{f_1^3 f_4^2 f_{12}^2}, \frac{f_1^3 f_4 f_6^7}{f_2^3 f_3 f_{12}^3}, \frac{f_3^3 f_4 f_6^5}{f_1 f_2 f_{12}^3}, \frac{f_1 f_4^2 f_6^{14}}{f_2^4 f_3^3 f_{12}^6}\right\}, \qquad \left\{\frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4}\right\}, \tag{3.1.11}$$

$$\left\{\frac{f_2^3 f_3^2}{f_1^2 f_6}, \frac{f_1^2 f_4^2 f_6^5}{f_2^3 f_3^2 f_{12}^2}\right\}, \qquad \left\{\frac{f_2^2 f_8 f_{12}^2}{f_4^2 f_{24}}, \frac{f_4^4 f_{12}^2}{f_2^2 f_8 f_{24}}\right\}, \tag{3.1.12}$$

$$\left\{\frac{f_2^6 f_3^2}{f_1^2 f_6^2}, \frac{f_1^2 f_4^2 f_6^4}{f_3^2 f_{12}^2}\right\}, \qquad \left\{\frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}\right\}, \tag{3.1.13}$$

$$\left\{\frac{f_2^7 f_3^3 f_{12}}{f_1 f_4^3 f_6^3}, \frac{f_1 f_2^4 f_6^6}{f_3^3 f_4^2 f_{12}^2}, \frac{f_1^3 f_2^5 f_{12}}{f_3 f_4^3 f_6}, \frac{f_2^{14} f_3 f_{12}^2}{f_1^3 f_4^6 f_6^4}\right\}, \qquad \left\{\frac{f_4^{15} f_6 f_{24}}{f_2^5 f_8^5 f_{12}^3}, \frac{f_2^5}{f_6}\right\}, \tag{3.1.14}$$

$$\left\{f_1 f_3, \frac{f_2^3 f_6^3}{f_1 f_3 f_4 f_{12}}\right\}, \qquad \left\{\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2}, \frac{f_4 f_6 f_8 f_{12}}{f_2 f_{24}}\right\}, \tag{3.1.15}$$

$$\left\{\frac{f_1 f_4^7 f_6^9}{f_2^5 f_3^3 f_{12}^5}, \frac{f_3^3 f_4^6}{f_1 f_2^2 f_{12}^2}, \frac{f_1^3 f_4^6 f_6^2}{f_2^4 f_3 f_{12}^2}, \frac{f_2^5 f_3 f_4^3}{f_1^3 f_6 f_{12}}\right\}, \qquad \left\{\frac{f_2^4 f_8^4 f_{12}^3}{f_4^3 f_6^2 f_{24}^2}, \frac{f_4^9 f_6^2}{f_2^4 f_{12}^3}\right\}, \tag{3.1.16}$$

$$\left\{\frac{f_1 f_4 f_6^8}{f_2^2 f_3^3 f_{12}^3}, \frac{f_2 f_3^3}{f_1 f_6}, \frac{f_1^3 f_6}{f_2 f_3}, \frac{f_2^8 f_3 f_{12}}{f_1^3 f_4^3 f_6^2}\right\}, \qquad \left\{\frac{f_2 f_8 f_{12}^2}{f_6 f_{24}}, \frac{f_4^3 f_6}{f_2 f_{12}}\right\}, \tag{3.1.17}$$

$$(3.1.18)\qquad \left\{\frac{f_2f_4^9}{f_1^2f_8^4},\frac{f_1^2f_4^{11}}{f_2^5f_8^4},\frac{f_2^9}{f_1^2f_4^3},\frac{f_1^2f_2^3}{f_4},\frac{f_1^6f_4}{f_2^3},\frac{f_2^{15}}{f_1^6f_4^5}\right\}, \qquad \left\{\frac{f_1^2f_8^{10}}{f_2f_4^3f_{16}^4},\frac{f_2^5f_8^{10}}{f_1^2f_4^5f_{16}^4}\right\},$$

$$(3.1.19)\qquad \left\{\frac{f_2^3f_3f_8f_{12}^8}{f_1f_4^3f_6^4f_{24}^3},\frac{f_1f_8f_{12}^7}{f_3f_4^2f_6f_{24}^3},\frac{f_1f_4^8f_6^3f_{24}}{f_2^4f_3f_8^3f_{12}^3},\frac{f_3f_4^7f_{24}}{f_1f_2f_8^3f_{12}^2},\right.$$

$$\left.\frac{f_1f_2^2f_6}{f_3f_4},\frac{f_2^5f_3f_{12}}{f_1f_4^2f_6^2},\frac{f_1f_4f_6^5}{f_2^2f_3f_{12}^2},\frac{f_2f_3f_6^2}{f_1f_{12}}\right\}, \qquad \left\{\frac{f_2^2f_3f_8^3f_{12}}{f_1f_4^2f_6f_{24}},\frac{f_1f_6^2f_8^3}{f_2f_3f_4f_{24}}\right\}.$$

Let $F(q)$ be any eta quotient in the collection on the left, and let $G(q)$ be any eta quotient in the corresponding collection on the right. Then

$$(3.1.20)\qquad F_{(0)} \subsetneq G_{(0)}.$$

Proof. Several of these claims follow for some pairs $(F(q), G(q))$ as a straightforward consequence of the remark at (2.4.1). In each case we prove the claim for just one eta quotient in each collection on the left and one eta quotient in the corresponding collection on the right. The result will hold generally if there are just two eta quotients in a collection, since the second will be the $q \to -q$ (or $q^2 \to -q^2$) partner of the first. In any collection with more than two members, all were shown to have identically vanishing coefficients in Theorem 3.1. For most proofs, we simply state which 2-dissection is used, and which eta quotient $C(q^2)$ it is multiplied by to produce the eta quotients $F(q)$ and $G(q)$, as described following (2.4.1) (the $F(q)$ so produced will be an eta quotient in the left collection and the $G(q)$ will be an eta quotient in the right collection).

We supply the details for (3.1.11). Recall (2.1.3):

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2f_{12}^3}{f_4f_6^2}.$$

If this identity is multiplied across by $C(q^2) = f_4f_6^7/(f_2^3f_{12}^3)$ one gets

$$\frac{f_1^3f_4f_6^7}{f_2^3f_3f_{12}^3} = \frac{f_4^4f_6^7}{f_2^3f_{12}^4} - 3q\frac{f_6^5}{f_2},$$

and thus clearly $F_{(0)} \subsetneq G_{(0)}$, if

$$F(q) = \frac{f_1^3f_4f_6^7}{f_2^3f_3f_{12}^3}, \qquad G(q) = \frac{f_4^4f_6^7}{f_2^3f_{12}^4}.$$

The full claim is true by the preceding remarks and Theorem 3.1, part (3.1.1).

The proof of (3.1.10) follows immediately from (2.1.9) (with $C(q^2) = 1$). The proof for the remaining parts of the theorem may be summarized thus:

- (3.1.12) follows from (2.1.13), with $C(q^2) = f_4^2f_6^5/(f_2^3f_{12}^2)$;
- (3.1.13) follows from (2.1.13), with $C(q^2) = f_4^2f_6^4/f_{12}^2$;
- (3.1.14) follows from (2.1.3), with $C(q^2) = f_2^5f_{12}/(f_4^3f_6)$ and finally Theorem 3.1, part (3.1.2);
- (3.1.15) follows immediately from (2.1.7) (with $C(q^2) = 1$);
- (3.1.16) follows from (2.1.3), with $C(q^2) = f_4^6f_6^2/(f_2^4f_{12}^2)$ and Theorem 3.1, part (3.1.3);
- (3.1.17) follows from (2.1.3), with $C(q^2) = f_6/f_2$ and Theorem 3.1, part (3.1.4);
- (3.1.18) follows from Lemma 2.15, with $C(q^4) = 1/f_4$ and Theorem 3.1, part (3.1.6);
- (3.1.19) follows from Lemma 2.16, with $C(q^4) = f_8f_{12}^7/(f_4^2f_{24}^3$ and Theorem 3.1, part (3.1.9). □

Remark: Parts (3.1.10) -(3.1.14) of Theorem 3.2 give some partial results on inclusion of vanishing coefficients between groups I and XI in Figure 1, while parts (3.1.15), (3.1.16) and (3.1.19), respectively, complete the proof of inclusion between, respectively, groups III and XIII, groups IV and XII and groups VIII and XIV. Parts (3.1.17) and (3.1.18), respectively, show partial inclusion results, respectively, between groups IX and XV and groups I and X. This does not show identical vanishing of coefficients of all eta quotients in group I, and this has likewise not been shown for all eta quotients in groups X and XV.

3.2. **Eta quotients with vanishing coefficient behaviour similar to f_1^6.** Some of the results suggested by Table 2 and Figure 2 were proved in [12]. We next use the methods of the present paper to prove some other such results. All the eta quotients in the next two theorems are from the list of eta quotients from which Table 2 and Figure 2 are derived (see [12] for the complete list).

Theorem 3.3. *Let $F(q)$ and $G(q)$ be any two eta quotients from one of the following lists.*

$$\left\{\frac{f_1f_4f_6^{10}}{f_2f_3^3f_{12}^4}, \frac{f_2^2f_3^3f_6}{f_1f_{12}}, \frac{f_1^3f_6^3}{f_3f_{12}}, \frac{f_2^9f_3}{f_1^3f_4^3}\right\}, \tag{3.2.1}$$

$$\left\{\frac{f_1f_4^2f_6^8}{f_2^3f_3^3f_{12}^3}, \frac{f_3^3f_4}{f_1f_6}, \frac{f_1^3f_4f_6}{f_2^2f_3}, \frac{f_2^7f_3f_{12}}{f_1^3f_4^2f_6^2}\right\}, \tag{3.2.2}$$

$$\left\{f_1^6, \frac{f_2^{18}}{f_1^6f_4^6}, \frac{f_3^{12}}{f_1^3f_9^3}, \frac{f_1^3f_4^3f_6^{36}f_9^3f_{36}^3}{f_2^9f_3^{12}f_{12}^{12}f_{18}^9}\right\}, \tag{3.2.3}$$

$$\left\{\frac{f_2^{10}f_3^2f_{12}^2}{f_1^4f_4^4f_6^4}, \frac{f_1^4f_6^2}{f_2^2f_3^2}, \frac{f_2^4}{f_1^2}, \frac{f_1^2f_4^2}{f_2^2}, \frac{f_2^4f_3f_{12}}{f_1f_4f_6^2}, \frac{f_1f_2f_6}{f_3}\right\}, \tag{3.2.4}$$

$$\left\{\frac{f_2^6f_8}{f_1^2f_4^3}, \frac{f_1^2f_8}{f_4}, \frac{f_1f_4^6f_6^3f_{24}}{f_2^3f_3f_8^2f_{12}^3}, \frac{f_3f_4^5f_{24}}{f_1f_8^2f_{12}^2}\right\}, \tag{3.2.5}$$

$$\left\{\frac{f_1f_4f_6^{13}}{f_2f_3^5f_{12}^5}, \frac{f_2^2f_3^5}{f_1f_6^2}, \frac{f_1^2f_3^4}{f_2f_6}, \frac{f_2^5f_6^{11}}{f_1^2f_3^4f_4^2f_{12}^4}\right\}, \tag{3.2.6}$$

$$\left\{\frac{f_1f_4f_6^{10}}{f_2f_3^3f_{12}^4}, \frac{f_2^2f_3^3f_6}{f_1f_{12}}, \frac{f_1^2f_3^2f_6^2}{f_2f_{12}}, \frac{f_2^5f_6^8}{f_1^2f_3^2f_4^2f_{12}^3}\right\}, \tag{3.2.7}$$

$$\left\{\frac{f_1f_3^5f_4}{f_2f_6^2}, \frac{f_2^2f_6^{13}}{f_1f_3^5f_{12}^5}, \frac{f_1^2f_6^{14}}{f_2f_3^6f_{12}^5}, \frac{f_2^5f_3^6f_{12}}{f_1^2f_4^2f_6^4}\right\}, \tag{3.2.8}$$

$$\left\{\frac{f_2^5f_3^2f_{12}^{11}}{f_1^2f_4^2f_6^6f_{24}^4}, \frac{f_1^2f_{12}^9}{f_2f_3^2f_{24}^4}, \frac{f_2^2f_{12}^9}{f_1f_3f_6f_{24}^4}, \frac{f_1f_3f_4f_{12}^{10}}{f_2f_6^4f_{24}^4}\right\}, \tag{3.2.9}$$

$$\left\{\frac{f_6^9f_8^2}{f_3^4f_4f_{24}^2}, \frac{f_3^4f_8^2f_{12}^4}{f_4f_6^3f_{24}^2}, \frac{f_4^2f_6^9}{f_3^4f_8f_{12}f_{24}}, \frac{f_3^4f_4^2f_{12}^3}{f_6^3f_8f_{24}}\right\}, \tag{3.2.10}$$

$$\left\{\frac{f_2^2f_3f_{12}^{13}}{f_1f_6^6f_{24}^5}, \frac{f_1f_4f_{12}^{12}}{f_2f_3f_6^3f_{24}^5}, \frac{f_2^5f_{12}^{13}}{f_1^2f_4^2f_6^5f_{24}^5}, \frac{f_1^2f_{12}^{13}}{f_2f_6^5f_{24}^5}\right\}, \tag{3.2.11}$$

$$\left\{\frac{f_1f_4f_6^7}{f_2f_3f_{12}^3}, \frac{f_2^2f_3f_6^4}{f_1f_{12}^2}, \frac{f_1^2f_6^5}{f_2f_{12}^2}, \frac{f_2^5f_6^5}{f_1^2f_4^2f_{12}^2}\right\}, \tag{3.2.12}$$

$$\left\{\frac{f_1f_3f_4}{f_2}, \frac{f_2^2f_6^3}{f_1f_3f_{12}}, \frac{f_1^2f_6^4}{f_2f_3^2f_{12}}, \frac{f_2^5f_3^2f_{12}}{f_1^2f_4^2f_6^2}\right\}, \tag{3.2.13}$$

$$\left\{\frac{f_2^2f_3f_{12}^3}{f_1f_6^2f_{24}}, \frac{f_1f_4f_6f_{12}^2}{f_2f_3f_{24}}, \frac{f_2^5f_{12}^3}{f_1^2f_4^2f_6f_{24}}, \frac{f_1^2f_{12}^3}{f_2f_6f_{24}}\right\}, \tag{3.2.14}$$

$$\left\{\frac{f_1f_4f_6^3}{f_2f_3f_{12}}, \frac{f_2^2f_3}{f_1}, \frac{f_1^2f_6}{f_2}, \frac{f_2^5f_6}{f_1^2f_4^2}\right\}, \tag{3.2.15}$$

$$\left\{\frac{f_8^2f_{12}^{14}}{f_4f_6^5f_{24}^6}, \frac{f_6^5f_8^2}{f_4f_{12}f_{24}}, \frac{f_4^2f_{12}^{13}}{f_6^5f_8f_{24}^5}, \frac{f_4^2f_6^5}{f_8f_{12}^2}\right\}, \tag{3.2.16}$$

$$\left\{\frac{f_4^{10}f_6f_{24}}{f_2^4f_8^4f_{12}^2}, \frac{f_2^4f_{12}}{f_4^2f_6}, f_2f_4, \frac{f_4^4}{f_2f_8}, \frac{f_2^2f_8^2f_{12}^2}{f_4^2f_6f_{24}}, \frac{f_4^4f_6}{f_2^2f_{12}}\right\}, \tag{3.2.17}$$

(3.2.18) $$\left\{\frac{f_8^2f_{12}^4}{f_4f_6f_{24}^2}, \frac{f_6f_8^2f_{12}}{f_4f_{24}}, \frac{f_4^2f_{12}^3}{f_6f_8f_{24}}, \frac{f_4^2f_6}{f_8}\right\},$$

(3.2.19) $$\left\{\frac{f_4^5f_6f_{24}}{f_2^2f_8^2f_{12}^2}, \frac{f_2^2f_{12}}{f_4f_6}, \frac{f_2f_8}{f_4}, \frac{f_4^2}{f_2}\right\},$$

(3.2.20) $$\left\{\frac{f_2^2f_4^7}{f_1^2f_8^3}, \frac{f_1^2f_4^9}{f_2^4f_8^3}, \frac{f_2^8}{f_1^2f_4^2}, f_1^2f_2^2\right\},$$

(3.2.21) $$\left\{\frac{f_2^4}{f_1^2}, \frac{f_1^2f_4^2}{f_2^2}, \frac{f_2^6f_8}{f_1^2f_4^3}, \frac{f_1^2f_8}{f_4}\right\},$$

(3.2.22) $$\left\{\frac{f_3f_4^9}{f_1f_2f_6f_8^3}, \frac{f_1f_4^{10}f_6^2}{f_2^4f_3f_8^3f_{12}}, \frac{f_2^5f_3}{f_1f_6}, \frac{f_1f_2^2f_4f_6^2}{f_3f_{12}}\right\},$$

(3.2.23) $$\left\{\frac{f_2^4f_3f_{12}}{f_1f_4f_6^2}, \frac{f_1f_2f_6}{f_3}, \frac{f_1f_4^6f_6^3f_{24}}{f_2^3f_3f_8^2f_{12}^3}, \frac{f_3f_4^5f_{24}}{f_1f_8^2f_{12}^2}\right\},$$

(3.2.24) $$\left\{\frac{f_1^2f_4^5f_6f_{24}}{f_2^3f_8^2f_{12}^2}, \frac{f_2^3f_4^3f_6f_{24}}{f_1^2f_8^2f_{12}^2}, \frac{f_1^2f_2f_{12}}{f_4f_6}, \frac{f_2^7f_{12}}{f_1^2f_4^3f_6}\right\},$$

(3.2.25) $$\left\{\frac{f_1f_6^2f_8}{f_3f_{12}}, \frac{f_2^3f_3f_8}{f_1f_4f_6}, \frac{f_1f_4^3f_6^2}{f_2^2f_3f_{12}}, \frac{f_2f_3f_4^2}{f_1f_6}\right\},$$

(3.2.26) $$\left\{\frac{f_4^3f_6^4}{f_3^2f_{24}}, \frac{f_3^2f_4^3f_{12}^2}{f_6^2f_{24}}, \frac{f_3^2f_4^3}{f_{12}}, \frac{f_4^3f_6^6}{f_3^2f_{12}^3}\right\},$$

(3.2.27) $$\left\{\frac{f_2^2f_3f_{12}^{13}}{f_1f_6^6f_{24}^5}, \frac{f_1f_4f_{12}^{12}}{f_2f_3f_6^3f_{24}^5}, \frac{f_1f_4f_6^7}{f_2f_3f_{12}^3}, \frac{f_2^2f_3f_6^4}{f_1f_{12}^2}\right\},$$

(3.2.28) $$\left\{\frac{f_2^5f_{12}^{13}}{f_1^2f_4^2f_6^5f_{24}^5}, \frac{f_1^2f_{12}^{13}}{f_2f_6^5f_{24}^5}, \frac{f_1^2f_6^5}{f_2f_{12}^2}, \frac{f_2^5f_6^5}{f_1^2f_4^2f_{12}^2}\right\},$$

(3.2.29) $$\left\{\frac{f_2^2f_3f_{12}^3}{f_1f_6^2f_{24}}, \frac{f_1f_4f_6f_{12}^2}{f_2f_3f_{24}}, \frac{f_1f_4f_6^3}{f_2f_3f_{12}}, \frac{f_2^2f_3}{f_1}\right\},$$

(3.2.30) $$\left\{\frac{f_2^5f_{12}^3}{f_1^2f_4^2f_6f_{24}}, \frac{f_1^2f_{12}^3}{f_2f_6f_{24}}, \frac{f_1^2f_6}{f_2}, \frac{f_2^5f_6}{f_1^2f_4^2}\right\},$$

(3.2.31) $$\left\{\frac{f_6^4f_{12}}{f_3^2f_{24}}, \frac{f_3^2f_{12}^3}{f_6^2f_{24}}, f_3^2, \frac{f_6^6}{f_3^2f_{12}^2}\right\}.$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. The various parts of this theorem are proved as follows:

- (3.2.1) by using (2.1.20) with $C(q^2) = f_2^9/f_4^3$;
- (3.2.2) by using (2.1.20) with $C(q^2) = f_2^7f_{12}/(f_4^2f_6^2)$;
- (3.2.3) follows directly from (2.2.12) (with $C(q^3) = 1$);
- (3.2.4) follows directly from (2.2.14) (with $C(q^3) = 1$);
- (3.2.5) follows directly from (2.2.11) (with $C(q^3) = 1$);
- (3.2.6) by using (2.2.10) with $C(q^3) = f_3^5/f_6^2$;
- (3.2.7) by using (2.2.10) with $C(q^3) = f_3^6f_6/f_{12}$;
- (3.2.8) by using (2.2.10) with $C(q^3) = f_6^{13}/(f_3^5f_{12}^5)$;
- (3.2.9) by using (2.2.10) with $C(q^3) = f_{12}^9/(f_3f_6f_{24}^4)$;
- (3.2.10) by using (2.2.19) with $C(q^3) = f_6^9/(f_3^4f_{24}^2)$;
- (3.2.11) by using (2.2.10) with $C(q^3) = f_3f_{12}^{13}/(f_6^6f_{24}^5)$;

- (3.2.12) by using (2.2.10) with $C(q^3) = f_3 f_6^4 / f_{12}^2$;
- (3.2.13) by using (2.2.10) with $C(q^3) = f_6^3 / (f_3 f_{12})$;
- (3.2.14) by using (2.2.10) with $C(q^3) = f_3 f_{12}^3 / (f_6^2 f_{24})$;
- (3.2.15) by using (2.2.10) with $C(q^3) = f_3$;
- (3.2.16) by using (2.2.18) with $C(q^3) = f_6^{14} / (f_3^5 f_{12}^6)$ and then applying the dilation $q \to q^2$;
- (3.2.17) by using (2.2.14) with $C(q^3) = f_3 / f_6$ and then applying the dilation $q \to q^2$;
- (3.2.18) by using (2.2.18) with $C(q^3) = f_6^4 / (f_3 f_{12}^2)$ and then applying the dilation $q \to q^2$;
- (3.2.19) by using (2.2.10) with $C(q^3) = 1$ and then applying the dilation $q \to q^2$;
- (3.2.20) follows directly from (2.3.64) (with $C(q^4) = 1$);
- (3.2.21) by using (2.3.63) with $C(q^4) = f_8 / f_4$;
- (3.2.22) by using (2.3.71) with $C(q^4) = f_4 / f_{12}$;
- (3.2.23) by using (2.3.65) with $C(q^4) = f_4^5 f_{24} / (f_8^2 f_{12}^2)$;
- (3.2.24) by using (2.3.72) with $C(q^4) = f_4^5 f_{24} / (f_8^2 f_{12}^2)$;
- (3.2.25) by using (2.3.73) with $C(q^4) = f_8 / f_{12}$;
- (3.2.26) by using (2.3.79) with $C(q^4) = f_4^3 / f_{12}$;
- (3.2.27) by using (2.3.74) with $C(q^4) = f_4 / f_{12}^3$;
- (3.2.28) by using (2.3.75) with $C(q^4) = f_{12}^{13} / f_{24}^5$;
- (3.2.29) by using (2.3.77) with $C(q^4) = f_4 f_{12}^2 / f_{24}$;
- (3.2.30) by using (2.3.76) with $C(q^4) = f_{12}^3 / f_{24}$;
- (3.2.31) follows directly from (2.3.79) (with $C(q^4) = 1$). □

Remark: We discuss briefly to what extent Theorem 3.3 has allowed us to prove what was suggested by experiment in terms of the claims in Table 2 and Figure 2.

The largest collection of eta quotients for which any of the theorems in the previous section enable us to show identical vanishing of coefficients is eight (Theorem 2.6), with most theorems showing identical vanishing of coefficients for collections of four eta quotients. Corollary 2.1 and Theorems 2.4 and 2.5 allow this to be done for collections of six eta quotients. Therefore, unless one of the collections tagged with a Roman numeral in Table 2 and Figure 2 is fairly small, it may take several of the theorems of the previous section to prove all the eta quotients in the collection have identically vanishing coefficients.

For example, (3.2.13) and (3.2.14) each show that two (disjoint) sets of four eta quotients from collection XIII in Table 2 and Figure 2 have identically vanishing coefficients (but not that two eta quotients from different sets have identically vanishing coefficients). Note that the set of four eta quotients (3.2.30), which has two eta quotients from (3.2.13) and two from (3.2.14), thus shows that all eight eta quotients in collection XIII have identically vanishing coefficients.

On the other hand, if a collection is too large, there may not be enough information to conclude identical vanishing of coefficients for all eta quotients in the collection. For example, all that Theorem 3.3 allows to conclude about collection I (which contains 42 eta quotients) is that there is one subset of 10 eta quotients with identically vanishing coefficients, and three other disjoint subsets of size 4, such that within each of the three subsets, the eta quotients have identically vanishing coefficients.

Overall, Theorem 3.3 provides a good deal of general information. Theorem 3.3 shows that all of the eta quotients in each of collections II, III, VII, VIII, IX, XII, XIII, XIV, XXI, XXII, XXIV and XXV have identically vanishing coefficients. Collections V, VI, XI, XVI, XVIII, XIX, XX and XXVII each have just two eta quotients, one being the $q \to -q$ partner of the other (in some cases, the $q^2 \to -q^2$ partner), so trivially all the eta quotients in each of these collections also have identically vanishing coefficients.

In addition there are partial results for some of the other collections. The claims for collection I have already been described. For the 16 eta quotients in collection IV, Theorem 3.3 shows the existence of three disjoint subsets, two of size 4 and one of size 6, with identical vanishing of coefficients for all eta quotients in each subset. For the 10 eta quotients in collection X, Theorem 3.3 shows that 8 of these have identically vanishing coefficients.

As was done with eta quotients with vanishing coefficient behaviour similar to f_1^4, we next prove some inclusion results.

Theorem 3.4. *Consider any one of the following pairs of collections of eta quotients* (3.2.32) - (3.2.40)*:*

(3.2.32)
$$\left\{\frac{f_2^{10}f_3^2f_{12}^2}{f_1^4f_4^4f_6^4}, \frac{f_1^4f_6^2}{f_2^2f_3^2}, \frac{f_2^4}{f_1^2}, \frac{f_1^2f_4^2}{f_2^2}, \frac{f_2^6f_8}{f_1^2f_4^3}, \frac{f_1^2f_8}{f_4}, \frac{f_2^4f_3f_{12}}{f_1f_4f_6^2}, \frac{f_1f_2f_6}{f_3}, \frac{f_1f_4^6f_6^3f_{24}}{f_2^3f_3f_8^2f_{12}^3}, \frac{f_3f_4^5f_{24}}{f_1f_8^2f_{12}^2}\right\} \qquad \left\{\frac{f_3^2f_{12}^2f_{18}^{10}}{f_6^4f_9^4f_{36}^4}, \frac{f_6^2f_9^4}{f_3^2f_{18}^2}\right\},$$

(3.2.33)
$$\left\{\frac{f_2^{11}}{f_1^4f_4^3}, \frac{f_1^4f_4}{f_2}\right\} \qquad \left\{\frac{f_2^3f_4^2}{f_8}, \frac{f_4^{11}}{f_2^3f_8^4}\right\},$$

(3.2.34)
$$\left\{\frac{f_2^2f_3^2f_4^4}{f_1^2f_6f_{12}}, \frac{f_1^2f_4^6f_6^5}{f_2^4f_3^2f_{12}^3}\right\} \qquad \left\{\frac{f_4^8f_{12}}{f_2^3f_8f_{24}}, \frac{f_2^3f_8^2f_{12}}{f_4f_{24}}\right\},$$

(3.2.35)
$$\left\{\frac{f_2^8f_6}{f_1f_3f_4^3}, \frac{f_1f_2^5f_3f_{12}}{f_4^2f_6^2}\right\} \qquad \left\{\frac{f_2^6f_8^2f_{12}^5}{f_4^4f_6^3f_{24}^2}, \frac{f_4^{14}f_6^3f_{24}}{f_2^6f_8^4f_{12}^4}\right\},$$

(3.2.36)
$$\left\{\frac{f_1f_4f_6^{10}}{f_2f_3^3f_{12}^4}, \frac{f_2^2f_3^3f_6}{f_1f_{12}}, \frac{f_1^2f_3^2f_6^2}{f_2f_{12}}, \frac{f_2^5f_6^8}{f_1^2f_3^2f_4^2f_{12}^3}, \frac{f_1^3f_6^3}{f_3f_{12}}, \frac{f_2^9f_3}{f_1^3f_4^3}\right\} \qquad \left\{\frac{f_4^3f_{12}^7}{f_6^3f_{24}^3}, \frac{f_4^3f_6^3}{f_{12}^2}\right\},$$

(3.2.37)
$$\left\{\frac{f_2^5f_3^2f_{12}^{11}}{f_1^2f_4^2f_6^6f_{24}^4}, \frac{f_1^2f_{12}^9}{f_2f_3^2f_{24}^4}, \frac{f_2^2f_{12}^9}{f_1f_3f_6f_{24}^4}, \frac{f_1f_3f_4f_{12}^{10}}{f_2f_6^4f_{24}^4}\right\} \qquad \left\{\frac{f_8^2f_{12}^{14}}{f_4f_6^5f_{24}^6}, \frac{f_6^5f_8^2}{f_4f_{12}f_{24}}, \frac{f_4^2f_{12}^{13}}{f_6^5f_8f_{24}^5}, \frac{f_4^2f_6^5}{f_8f_{12}^2}\right\},$$

(3.2.38)
$$\left\{\frac{f_6^9f_8^2}{f_3^4f_4f_{24}^2}, \frac{f_3^4f_8^2f_{12}^4}{f_4f_6^3f_{24}^2}, \frac{f_4^2f_6^9}{f_3^4f_8f_{12}f_{24}}, \frac{f_3^4f_4^2f_{12}^3}{f_6^3f_8f_{24}}\right\} \qquad \left\{\frac{f_8^2f_{12}^{14}}{f_4f_6^5f_{24}^6}, \frac{f_6^5f_8^2}{f_4f_{12}f_{24}}, \frac{f_4^2f_{12}^{13}}{f_6^5f_8f_{24}^5}, \frac{f_4^2f_6^5}{f_8f_{12}^2}\right\},$$

(3.2.39)
$$\left\{\frac{f_1f_3f_4}{f_2}, \frac{f_2^2f_6^3}{f_1f_3f_{12}}, \frac{f_1^2f_6^4}{f_2f_3^2f_{12}}, \frac{f_2^5f_3^2f_{12}}{f_1^2f_4^2f_6^2}\right\} \qquad \left\{\frac{f_8^2f_{12}^4}{f_4f_6f_{24}^2}, \frac{f_6f_8^2f_{12}}{f_4f_{24}}, \frac{f_4^2f_{12}^3}{f_6f_8f_{24}}, \frac{f_4^2f_6}{f_8}\right\},$$

(3.2.40)
$$\left\{\frac{f_1f_4^2f_6^8}{f_2^3f_3^3f_{12}^3}, \frac{f_3^3f_4}{f_1f_6}, \frac{f_1^3f_4f_6}{f_2^2f_3}, \frac{f_2^7f_3f_{12}}{f_1^3f_4^2f_6^2}\right\} \qquad \left\{\frac{f_4^{10}f_6f_{24}}{f_2^4f_8^4f_{12}^2}, \frac{f_2^4f_{12}}{f_4^2f_6}, f_2f_4, \frac{f_4^4}{f_2f_8}, \frac{f_2^2f_8^2f_{12}^2}{f_4^2f_6f_{24}}, \frac{f_4^4f_6}{f_2^2f_{12}}\right\},$$

(3.2.41)
$$\left\{\frac{f_3^4f_6}{f_{12}}, \frac{f_6^{13}}{f_3^4f_{12}^5}\right\} \qquad \left\{\frac{f_6f_{12}^6}{f_{24}^3}, \frac{f_{12}^9}{f_6f_{24}^4}\right\}.$$

Let $F(q)$ be any eta quotient in the collection on the left, and let $G(q)$ be any eta quotient in the corresponding collection on the right. Then

(3.2.42)
$$F_{(0)} \subsetneqq G_{(0)}.$$

Proof. For each pair, we prove the statement for just one eta quotient in each collection, since all the eta quotients in each collection have identically vanishing coefficients, either by Theorem 3.3 or because there are just two eta quotients in a collection, each being the $q \to -q$ or $q^2 \to -q^2$ partner of the other.

As with Theorem 3.2, several proofs for particular pairs $(F(q), G(q))$ are a straightforward consequences of the remark at (2.4.1). We use the notation of the remark at (2.4.1) and simply state which dissection identity is used; the form of the function $C(q^2)$; and which parts of Theorem 3.3 give the full result. The mechanics of this type of proof have been described in some detail in the proof of Theorem 3.2. The proofs of the various parts may be summarized as follows:

- (3.2.32) follows directly from (2.2.17), together Theorem 3.3, parts (3.2.4), (3.2.5) and (3.2.21);
- (3.2.33) follows from (2.1.9) with $C(q^2) = f_4/f_2$;
- (3.2.34) follows from (2.1.13) with $C(q^2) = f_4^6 f_6^5/(f_2^4 f_{12}^3)$;
- (3.2.35) follows from (2.1.7) with $C(q^2) = f_2^5 f_{12}/(f_4^2 f_6^2)$;
- (3.2.36) follows from (2.1.5) with $C(q^2) = f_2^2 f_6/f_{12}$ and Theorem 3.3, parts (3.2.1) and (3.2.7);
- (3.2.37) follows from (2.1.13) with $C(q^2) = f_{12}^9/(f_2 f_{24}^4)$ and Theorem 3.3, parts (3.2.9) and (3.2.16);
- (3.2.38) follows from (2.1.9), after first applying a dilation $q \to q^3$, then multiplying across by $C(q^2) = f_8^2 f_{12}^4/(f_4 f_6^3 f_{24}^2)$, and finally using Theorem 3.3, parts (3.2.10) and (3.2.16);
- (3.2.39) follows from (2.1.7) with $C(q^2) = f_4/f_2$ and Theorem 3.3, parts (3.2.13) and (3.2.18);
- (3.2.40) follows from (2.1.5) with $C(q^2) = f_4/f_6$ and Theorem 3.3, parts (3.2.2) and (3.2.17);
- (3.2.41) follows from (2.1.9), after first applying a dilation $q \to q^3$, then multiplying across by $C(q^2) = f_6/f_{12}$. □

Remark: Parts (3.2.37)-(3.2.40) of Theorem 3.4, respectively complete the proof of inclusion (as indicated by arrows in Figure 2) between the following collections of eta quotients from Table 2: VIII and XXI; IX and XXI; XII and XXIV; and XIV and XXII. In addition, the other parts of Theorem 3.4 show partial inclusion results between collections I and XVII; I and XV; as well as XVII and XXVI.

3.3. **Eta quotients with vanishing coefficient behaviour similar to f_1^8.** As was done with f_1^4 and f_1^6, we summarize what experiment suggests (see [12]) about the collections of eta quotients with vanishing coefficient behaviour similar to f_1^8 in the following table and graph.

Table 3: Eta quotients with vanishing behaviour similar to f_1^8

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	24	II †	2
III †	2	IV	60
V †	2	VI	6
VII †	2	VIII	4
IX †	2	X †	2
XI *	4	XII *	4
XIII *	4	XIV	4
XV †	2	XVI †	2
XVII †	2	XVIII †	2
XIX	6	XX †	2
XXI †	2	XXII †	4
XXIII †	2	XXIV	4
XXV †	6		

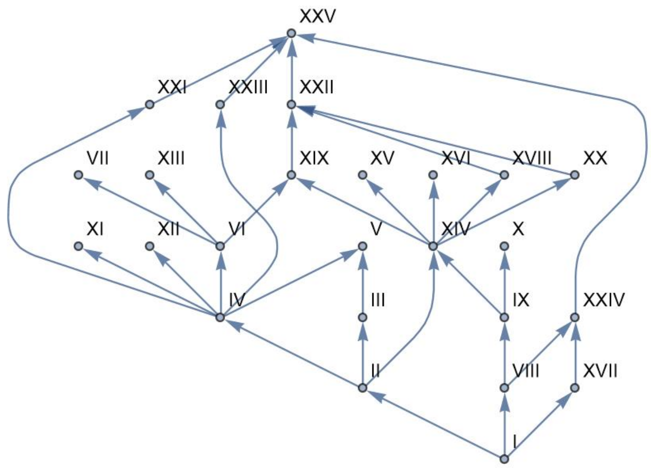

FIGURE 3. The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^8

At the end of Section 1, we remarked that if Table 1 and Figure 1 represented the true situation for eta quotients with vanishing coefficient behaviour similar to f_1^4, then this table and graph are embedded in their entirety, via the dilation $q \to q^2$ in the corresponding Table 3 and Figure 3 above. We did indeed discover experimental evidence for this. This correspondence is summarized in the following table.

Table 4: Groups of eta quotients in Table 3 that arise as $q \to q^2$ dilations of groups of eta quotients in Table 1

Table 1 Group (f_1^4)	Table 3 Group (f_1^8)	Table 1 Group (f_1^4)	Table 3 Group (f_1^8)
I	IV	III	V
IV	VI	V	VII
VII	XI	VIII	XII
IX	XIII	XI	XIX
XVI	XXI	XVII	XXII
XVIII	XXIII	XIX	XXV

Some comments are in order.

(1) Not all of the groups in Table 1 are represented via their $q \to q^2$ dilations in Table 3. This is because the eta quotients formed via the $q \to q^2$ dilation lay outside the range of the search performed and described in [12].
(2) Likewise, the reason some of the groups in Table 1 are larger than the corresponding groups in Table 3 is that some of the eta quotients formed via the $q \to q^2$ dilation also lay outside the range of this search.
(3) All of the results in Table 3 with a * were derived via this $q \to q^2$ dilation of results in Table 1, as may be easily seen from Table 4.

To be more precise about item (3) above, the proof that all of the eta quotients in each of groups XI, XII and XIII from Table 3 have identically vanishing coefficients may be justified by applying a $q \to q^2$ dilation to the collections of eta quotients at (3.1.8), (3.1.9) and (3.1.4), respectively. Such analysis provides the proof of identically vanishing coefficients in each of groups, VII, VIII and IX, respectively, from Table 1.

Apart from what has been described above, in this subsection we confine ourselves to deriving some results that may not be derived via dilation. The next result concerns two groups of four eta quotients each from group I of Table 3.

Theorem 3.5. *Let $F(q)$ and $G(q)$ be any two eta quotients from either of the following lists.*

$$\left\{\frac{f_2^7 f_3}{f_1^3 f_6}, \frac{f_1^3 f_4^3 f_6^2}{f_2^2 f_3 f_{12}}, \frac{f_3^3 f_4^3}{f_1 f_{12}}, \frac{f_1 f_4^4 f_6^9}{f_2^3 f_3^3 f_{12}^4}\right\}, \tag{3.3.1}$$

$$\left\{\frac{f_2^7 f_3 f_{12}}{f_1^3 f_4^3 f_6}, \frac{f_1^3 f_6^2}{f_2^2 f_3}, \frac{f_3^3}{f_1}, \frac{f_1 f_4 f_6^9}{f_2^3 f_3^3 f_{12}^3}\right\}. \tag{3.3.2}$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. Each of (3.3.1) and (3.3.2), respectively, follow from (2.1.20) upon setting, respectively, $C(q^2) = f_2^7/f_6$ and $C(q^2) = f_2^7 f_{12}/(f_4^3 f_6)$. □

The following inclusion results are, in part, elementary consequences of some of the 2-dissections listed in Lemma 2.2 and the remark at (2.4.1). However, they do provide further supporting evidence for the connections between the various groups of eta quotients in Table 3, which experiment suggests are connected as in Figure 3. Many of the eta quotients in Table 3 derive from the $q \to q^2$ dilation of eta quotients in Table 1, and perhaps it is not surprising that some of these are the even part of other eta quotients in Table 3.

Theorem 3.6. *Consider any one of the following pairs of collections of eta quotients* (3.3.3) - (3.3.13)*:*

$$\left\{\frac{f_2^7 f_3}{f_1^3 f_6}, \frac{f_1^3 f_4^3 f_6^2}{f_2^2 f_3 f_{12}}, \frac{f_3^3 f_4^3}{f_1 f_{12}}, \frac{f_1 f_4^4 f_6^9}{f_2^3 f_3^3 f_{12}^4}\right\}, \qquad \left\{\frac{f_2^2 f_8^2 f_{12}^4}{f_6^2 f_{24}^2}, \frac{f_4^6 f_6^2}{f_2^2 f_{12}^2}\right\}, \tag{3.3.3}$$

$$\left\{\frac{f_2^7 f_3 f_{12}}{f_1^3 f_4^3 f_6}, \frac{f_1^3 f_6^2}{f_2^2 f_3}, \frac{f_3^3}{f_1}, \frac{f_1 f_4 f_6^9}{f_2^3 f_3^3 f_{12}^3}\right\}, \qquad \left\{\frac{f_2^2 f_8^2 f_{12}^5}{f_4^3 f_6^2 f_{24}^2}, \frac{f_4^3 f_6^2}{f_2^2 f_{12}}\right\}, \tag{3.3.4}$$

$$\left\{\frac{f_2^4 f_6^2}{f_1 f_3 f_4^2}, \frac{f_1 f_2 f_3 f_{12}}{f_4 f_6}\right\}, \qquad \left\{\frac{f_2^2 f_8^2 f_{12}^5}{f_4^3 f_6^2 f_{24}^2}, \frac{f_4^3 f_6^2}{f_2^2 f_{12}}\right\}, \tag{3.3.5}$$

$$\left\{\frac{f_2^4 f_4 f_6^2}{f_1 f_3 f_{12}}, \frac{f_1 f_2 f_3 f_4^2}{f_6}\right\}, \qquad \left\{\frac{f_2^2 f_8^2 f_{12}^4}{f_6^2 f_{24}^2}, \frac{f_4^6 f_6^2}{f_2^2 f_{12}^2}\right\}, \tag{3.3.6}$$

$$\left\{\frac{f_1^4 f_4^2}{f_2^2}, \frac{f_2^{10}}{f_1^4 f_4^2}\right\}, \qquad \left\{f_2^4, \frac{f_4^{12}}{f_2^4 f_8^4}\right\}, \tag{3.3.7}$$

$$\left\{\frac{f_1^2 f_4 f_6^4}{f_2^2 f_3^2 f_{12}}, \frac{f_2^4 f_3^2 f_{12}}{f_1^2 f_4 f_6^2}\right\}, \qquad \left\{\frac{f_4^3 f_{12}^3}{f_2 f_6 f_8 f_{24}}, f_2 f_6\right\}, \tag{3.3.8}$$

$$\left\{\frac{f_1^2 f_4^4}{f_2 f_8}, \frac{f_2^5 f_4^2}{f_1^2 f_8}\right\}, \qquad \left\{\frac{f_8^{10}}{f_4^2 f_{16}^4}, \frac{f_4^2 f_8^4}{f_{16}^2}\right\}, \tag{3.3.9}$$

$$\left\{\frac{f_2^5 f_8^5}{f_1^2 f_4^2 f_{16}^2}, \frac{f_1^2 f_8^5}{f_2 f_{16}^2}\right\}, \qquad \left\{\frac{f_8^{10}}{f_4^2 f_{16}^4}, \frac{f_4^2 f_8^4}{f_{16}^2}\right\}, \tag{3.3.10}$$

$$\left\{\frac{f_2^5 f_{16}^3}{f_1^2 f_4^2 f_8 f_{32}}, \frac{f_1^2 f_{16}^3}{f_2 f_8 f_{32}}\right\}, \qquad \left\{\frac{f_8^4 f_{16}}{f_4^2 f_{32}}, \frac{f_4^2 f_{16}^3}{f_8^2 f_{32}}, \frac{f_8^6}{f_4^2 f_{16}^2}, f_4^2\right\}, \tag{3.3.11}$$

$$\left\{\frac{f_1^2 f_8}{f_2}, \frac{f_2^5 f_8}{f_1^2 f_4^2}\right\}, \qquad \left\{\frac{f_8^4 f_{16}}{f_4^2 f_{32}}, \frac{f_4^2 f_{16}^3}{f_8^2 f_{32}}, \frac{f_8^6}{f_4^2 f_{16}^2}, f_4^2\right\}, \tag{3.3.12}$$

$$\text{(3.3.13)} \qquad \left\{\frac{f_1^2 f_4^2}{f_2}, \frac{f_2^5}{f_1^2}\right\}, \qquad\qquad \left\{\frac{f_{16}^{13}}{f_8^5 f_{32}^5}, \frac{f_8^5}{f_{16}^2}, \frac{f_{16}^3}{f_8 f_{32}}, f_8\right\}.$$

Let $F(q)$ be any eta quotient in the collection on the left, and let $G(q)$ be any eta quotient in the corresponding collection on the right. Then

$$F_{(0)} \subsetneqq G_{(0)}. \tag{3.3.14}$$

Proof. Once again it is enough in each case to prove each statement for one choice of eta quotient for $F(q)$ and likewise for one choice for $G(q)$, since all the eta quotients in each collection have identically vanishing coefficients. This follows either from what has already been said about the groups in Table 3 labelled with either a * or †, or from Theorem 3.5. As in the proof of Theorem 3.4, with the notation of the remark at (2.4.1), we limit the proof to stating which dissection identity is used; the form of the function $C(q^2)$; and possibly which parts of Theorem 3.5 give the full result. The proofs of the various parts of Theorem 3.6 may be summarized as follows:
- (3.3.3) follows from (2.1.4) with $C(q^2) = f_4^3/f_{12}$ and (3.3.1);
- (3.3.4) follows from (2.1.4) with $C(q^2) = 1$ and (3.3.2);
- (3.3.5) follows from (2.1.7) with $C(q^2) = f_2 f_{12}/(f_4 f_6)$;
- (3.3.6) follows from (2.1.7) with $C(q^2) = f_2 f_4^2/f_6$;
- (3.3.7) follows from (2.1.9) with $C(q^2) = f_4^2/f_2^2$;
- (3.3.8) follows from (2.1.13) with $C(q^2) = f_4 f_6^4/(f_2^2 f_{12})$;
- (3.3.9) follows from (2.1.2) with $C(q^2) = f_4^4/f_8$;
- (3.3.10) follows from (2.1.1) with $C(q^2) = f_8^5/f_{16}^2$;
- (3.3.11) follows from (2.1.1) with $C(q^2) = f_{16}^3/(f_8 f_{12})$ and [12, Corollary 8.2];
- (3.3.12) follows from (2.1.2) with $C(q^2) = f_8$ and [12, Corollary 8.2];
- (3.3.13) follows from (2.1.2) with $C(q^2) = f_4^2$ and [12, Corollary 8.2]. □

Remark: The inclusion statements represented by (3.3.3) - (3.3.6) prove partial inclusion results between group I and group IV; (3.3.7) shows partial inclusion between group II and group IV; (3.3.8) completes the proof of proving inclusion between group III and group V; (3.3.9) and (3.3.10) prove partial inclusion results between groups IX and XIV and group XIX; (3.3.11) completes the proof of proving inclusion between group XVIII and group XXII; (3.3.12) completes the proof of proving inclusion between group XX and group XXII; and (3.3.13) shows partial inclusion between group XXIV and group XXV.

3.4. Eta quotients with vanishing coefficient behaviour similar to f_1^{10}.

What the experiments in [12] suggested about eta quotients with vanishing coefficient behaviour similar to f_1^{10} is summarized in the following table and graph, where the * and † symbols and arrows have the same meaning as in similar tables and graphs elsewhere in the paper.

Table 5: Eta quotients with vanishing behaviour similar to f_1^{10}

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	38	II *	4
III †	2	IV *	4
V	4	VI †	2
VII	6	VIII †	2
IX *	4	X †	2
XI *	4	XII †	2
XIII †	2	XIV †	2
XV †	2	XVI †	2

XVII	8	XVIII †	2
XIX *	4	XX †	2
XXI †	2	XXII †	2
XXIII	4	XXIV †	4
XXV †	6		

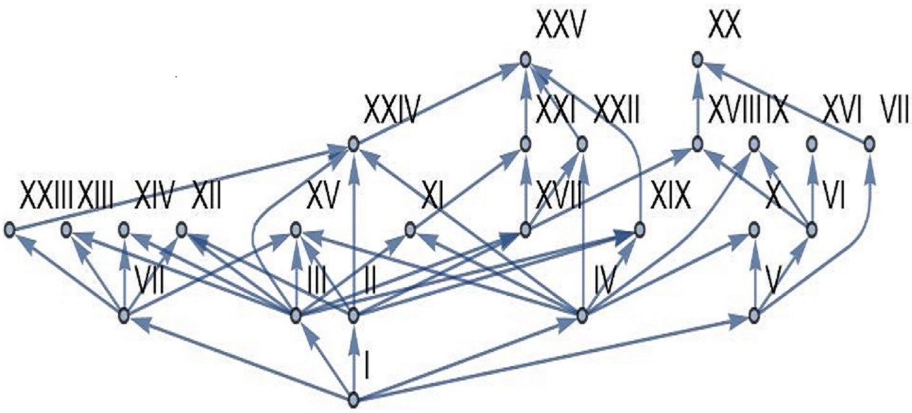

FIGURE 4. The grouping of eta-quotients in Table 5, which have vanishing coefficient behaviour similar to f_1^{10}

We next show that various collections of eta quotients from Table 5 and Figure 4 have identically vanishing coefficients.

Theorem 3.7. *Let $F(q)$ and $G(q)$ be any two eta quotients from any one of the following lists.*

$$\left\{\frac{f_2^{10}f_3f_{12}}{f_1^3f_4^4f_6}, \frac{f_1^3f_2f_6^2}{f_3f_4}, \frac{f_2^3f_3^3}{f_1f_4}, \frac{f_1f_6^9}{f_3^3f_{12}^3}\right\}, \tag{3.4.1}$$

$$\left\{\frac{f_4^{13}}{f_1^2f_8^5}, \frac{f_1^2f_4^{15}}{f_2^6f_8^5}, \frac{f_2^{10}}{f_1^2f_4^2}, f_1^2f_2^4\right\}, \tag{3.4.2}$$

$$\left\{\frac{f_2^4f_4^3}{f_1^2f_8}, \frac{f_1^2f_4^5}{f_2^2f_8}, \frac{f_2^6}{f_1^2}, f_1^2f_4^2\right\}, \tag{3.4.3}$$

$$\left\{\frac{f_2f_3f_6}{f_1}, \frac{f_1f_4f_6^4}{f_2^2f_3f_{12}}, \frac{f_2^3f_3f_8f_{12}^6}{f_1f_4^3f_6^3f_{24}^2}, \frac{f_1f_8f_{12}^5}{f_3f_4^2f_{24}^2}\right\}, \tag{3.4.4}$$

$$\left\{\frac{f_2^6f_8f_{12}^5}{f_1^2f_4^2f_6^2f_{24}^2}, \frac{f_1^2f_8f_{12}^5}{f_6^2f_{24}^2}, \frac{f_1^2f_4^3f_6^2}{f_2^2f_{12}}, \frac{f_2^4f_4f_6^2}{f_1^2f_{12}}\right\}, \tag{3.4.5}$$

$$\left\{\frac{f_3f_4^{12}f_{12}}{f_1f_2^3f_6f_8^5}, \frac{f_1f_4^{13}f_6^2}{f_2^6f_3f_8^5}, \frac{f_2^7f_3f_{12}}{f_1f_4^3f_6}, \frac{f_1f_2^4f_6^2}{f_3f_4^2}\right\}, \tag{3.4.6}$$

$$\left\{\frac{f_2f_3f_4^2f_{12}}{f_1f_6f_8}, \frac{f_1f_4^3f_6^2}{f_2^2f_3f_8}, \frac{f_1f_6^2}{f_3}, \frac{f_2^3f_3f_{12}}{f_1f_4f_6}\right\}, \tag{3.4.7}$$

$$\left\{\frac{f_1f_4f_6^7}{f_2^2f_3^3f_{12}^2}, \frac{f_2f_3^3f_{12}}{f_1f_6^2}, \frac{f_1^3f_{12}}{f_2f_3}, \frac{f_2^8f_3f_{12}^2}{f_1^3f_4^3f_6^3}\right\}, \tag{3.4.8}$$

$$\left\{\frac{f_1^2f_{20}^3}{f_2f_{10}f_{40}}, \frac{f_2^5f_{20}^3}{f_1^2f_4^2f_{10}f_{40}}, \frac{f_1^2f_{10}}{f_2}, \frac{f_2^5f_{10}}{f_1^2f_4^2}\right\}. \tag{3.4.9}$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. Once again the proofs of the various statements are straightforward consequences of what has been shown in Section 2:
- (3.4.1) by using (2.1.20) with $C(q^2) = f_2^{10} f_{12}/(f_4^4 f_6)$;
- (3.4.2) by using (2.3.86) with $C(q^4) = 1$;
- (3.4.3) by using (2.3.85) with $C(q^4) = f_4^2$;
- (3.4.4) by using (2.3.66) with $C(q^4) = f_8 f_{12}^5/(f_4^2 f_{24}^2)$;
- (3.4.5) by using (2.3.67) with $C(q^4) = f_8 f_{12}^5/f_{24}^2$;
- (3.4.6) by using (2.3.68) with $C(q^4) = f_4^{12} f_{12}/f_8^5$;
- (3.4.7) by using (2.3.73) with $C(q^4) = 1$;
- (3.4.8) by using (2.1.20) with $C(q^2) = f_2^8 f_{12}^2/(f_4^3 f_6^3)$;
- (3.4.9) by using (2.3.32) with $m = 5$. □

Remark: The collections of eta quotients in collections (3.4.1) - (3.4.4) are all in group I of Table 5 / Figure 4. The collections (3.4.5) - (3.4.9) comprise respectively, groups II, IV, IX, XI and XIX, (so that here also our theorem has completed the task of showing identical vanishing of coefficients within all of these latter groups).

We next prove some inclusion results that support the relationship between the various groups of eta quotients in Table 5, as indicated by the arrows in Figure 4.

Theorem 3.8. *Consider any one of the following pairs of collections of eta quotients* (3.3.3)-(3.3.13)*:*

(3.4.10) $$\left\{\frac{f_2^9}{f_1^4 f_4}, \frac{f_1^4 f_4^3}{f_2^3}\right\}, \qquad \left\{\frac{f_2^5 f_8}{f_4^2}, \frac{f_4^{13}}{f_2^5 f_8^4}\right\},$$

(3.4.11) $$\left\{\frac{f_2^{10} f_3 f_{12}}{f_1^3 f_4^4 f_6}, \frac{f_1^3 f_2 f_6^2}{f_3 f_4}, \frac{f_2^3 f_3^3}{f_1 f_4}, \frac{f_1 f_6^9}{f_3^3 f_{12}^3}\right\}, \qquad \left\{\frac{f_4^5 f_{12}^5}{f_2 f_6^2 f_8 f_{24}^2}, \frac{f_2 f_4^2 f_6^2}{f_{12}}\right\},$$

(3.4.12) $$\left\{\frac{f_1 f_3 f_4 f_6^5}{f_2^2 f_{12}^2}, \frac{f_2 f_6^8}{f_1 f_3 f_{12}^3}\right\}, \qquad \left\{\frac{f_2 f_8^3 f_{12}^{14}}{f_4^4 f_6^4 f_{24}^6}, \frac{f_6^4 f_8^2 f_{12}^2}{f_2 f_4 f_{24}^2}\right\},$$

(3.4.13) $$\left\{\frac{f_2^{11} f_8^2 f_{12}^5}{f_1^4 f_4^6 f_6^2 f_{24}^2}, \frac{f_1^4 f_8^2 f_{12}^5}{f_2 f_4^2 f_6^2 f_{24}^2}\right\}, \qquad \left\{\frac{f_4^8 f_{12}^5}{f_2^3 f_6^2 f_8^2 f_{24}^2}, \frac{f_2^3 f_6^2 f_8}{f_4 f_{12}}\right\},$$

(3.4.14) $$\left\{\frac{f_1 f_3 f_4^{10} f_{12}}{f_2^4 f_6 f_8^4}, \frac{f_4^9 f_6^2}{f_1 f_2 f_3 f_8^4}\right\}, \qquad \left\{\frac{f_4^8 f_{12}^5}{f_2^3 f_6^2 f_8^2 f_{24}^2}, \frac{f_2^3 f_6^2 f_8}{f_4 f_{12}}\right\},$$

(3.4.15) $$\left\{\frac{f_1 f_3 f_{12}}{f_6}, \frac{f_2^3 f_6^2}{f_1 f_3 f_4}\right\}, \qquad \left\{\frac{f_2 f_8^2 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}, \frac{f_4 f_6^2 f_8}{f_2 f_{12}}\right\},$$

(3.4.16) $$\left\{\frac{f_1 f_4 f_6^7}{f_2^2 f_3^3 f_{12}^2}, \frac{f_2 f_3^3 f_{12}}{f_1 f_6^2}, \frac{f_1^3 f_{12}}{f_2 f_3}, \frac{f_2^8 f_3 f_{12}^2}{f_1^3 f_4^3 f_6^3}\right\}, \qquad \left\{f_2 f_8, \frac{f_4^3}{f_2}\right\}.$$

Let $F(q)$ be any eta quotient in the collection on the left, and let $G(q)$ be any eta quotient in the corresponding collection on the right. Then

(3.4.17) $$F_{(0)} \subsetneqq G_{(0)}.$$

Proof. As in the proof of similar theorems in the paper, it is enough to prove each statement for one choice of eta quotient $F(q)$ and likewise one choice for $G(q)$, since all the eta quotients in each collection have identically vanishing coefficients. The proofs of the various parts of Theorem 3.8 may be summarized as follows:
- (3.4.10) follows from (2.1.9) with $C(q^2) = f_4^3/f_2^3$;

- (3.4.11) follows from (2.1.5)with $C(q^2) = f_2^3/f_4$ and (3.4.1);
- (3.4.12) follows from (2.1.7) with $C(q^2) = f_4 f_6^5/(f_2^2 f_{12}^2)$;
- (3.4.13) follows from (2.1.9) with $C(q^2) = f_8^2 f_{12}^5/(f_2 f_4^2 f_6^2 f_{24}^2)$;
- (3.4.14) follows from (2.1.7) with $C(q^2) = f_4^{10} f_{12}/(f_2^4 f_6 f_8^4)$;
- (3.4.15) follows from (2.1.7) with $C(q^2) = f_{12}/f_6$;
- (3.4.16) follows from (2.1.3) with $C(q^2) = f_{12}/f_2$ and (3.4.8). □

We note that (3.4.10) and (3.4.11) provide some partial evidence for the inclusion of group I in group XVII. Note that (3.4.12) likewise supports the experimental evidence for the inclusion of group III in group XVII. Together, (3.4.13) and (3.4.14) complete the proof that group V is included in group XVIII; (3.4.15) likewise shows that group VIII is included in group XX; and (3.4.16) completes the proof that group XI is included in group XXI.

3.5. **Eta quotients with vanishing coefficient behaviour similar to f_1^{14}.** Once again, we begin by summarizing what the experiments whose output was described in [12] suggested about eta quotients with vanishing coefficient behaviour similar to f_1^{14}, where the * and † symbols and arrows have the same meaning as elsewhere in the paper.

Table 6: Eta quotients with vanishing behaviour similar to f_1^{14}

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	32	II *	4
III *	4	IV *	4
V †	2	VI	12
VII *	4	VIII	8
IX †	2	X †	2
XI †	2	XII †	2
XIII †	2	XIV †	4
XV †	6		

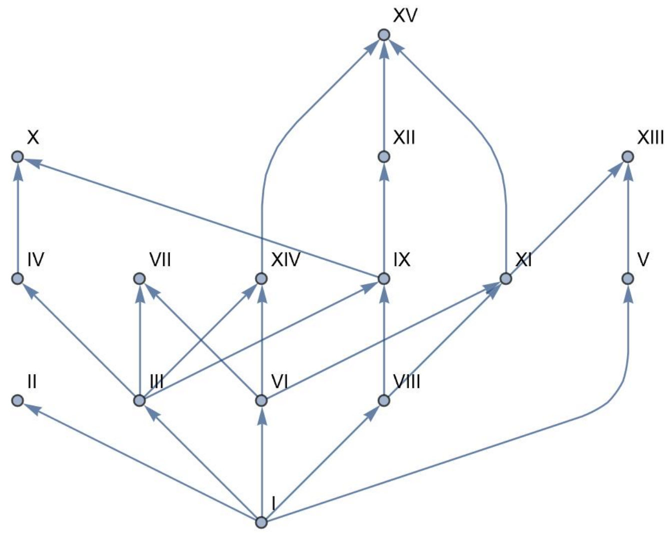

FIGURE 5. The grouping of eta-quotients in Table 6, which have vanishing coefficient behaviour similar to f_1^{14}

As elsewhere, dissection methods are next used to show that various collections of eta quotients from Table 6 and Figure 5 have identically vanishing coefficients.

Theorem 3.9. *Let $F(q)$ and $G(q)$ be any two eta quotients from any one of the following lists.*

$$\left\{\frac{f_2^2f_4^9}{f_1^2f_8^3},\frac{f_1^2f_4^{11}}{f_2^4f_8^3},\frac{f_2^8}{f_1^2},f_1^2f_2^2f_4^2\right\}, \tag{3.5.1}$$

$$\left\{\frac{f_2^4f_4^2}{f_1^2},\frac{f_1^2f_4^4}{f_2^2},\frac{f_2^6f_8}{f_1^2f_4},f_1^2f_4f_8\right\}, \tag{3.5.2}$$

$$\left\{\frac{f_2^7f_{12}}{f_1^2f_4f_6},\frac{f_1^2f_2f_4f_{12}}{f_6},\frac{f_2^3f_4^5f_6f_{24}}{f_1^2f_8^2f_{12}^2},\frac{f_1^2f_4^7f_6f_{24}}{f_2^3f_8^2f_{12}^2}\right\}, \tag{3.5.3}$$

$$\left\{\frac{f_2f_3f_4f_{12}}{f_1f_6},\frac{f_1f_4^2f_6^2}{f_2^2f_3},\frac{f_1f_6^2f_8}{f_3f_4},\frac{f_2^3f_3f_8f_{12}}{f_1f_4^2f_6}\right\}, \tag{3.5.4}$$

$$\left\{\frac{f_2^5f_3f_{12}}{f_1f_4f_6},\frac{f_1f_2^2f_6^2}{f_3},\frac{f_1f_4^9f_6^2}{f_2^4f_3f_8^3},\frac{f_3f_4^8f_{12}}{f_1f_2f_6f_8^3}\right\}, \tag{3.5.5}$$

$$\left\{\frac{f_3f_4^4f_{24}}{f_1f_8^2f_{12}},\frac{f_1f_4^5f_6^3f_{24}}{f_2^3f_3f_8^2f_{12}^2},\frac{f_2^4f_3f_{12}^2}{f_1f_4^2f_6^2},\frac{f_1f_2f_6f_{12}}{f_3f_4}\right\}, \tag{3.5.6}$$

$$\left\{\frac{f_2^7f_3f_{12}^{10}}{f_1^3f_4f_6^6f_8f_{24}^3},\frac{f_1^3f_4^2f_{12}^9}{f_2^2f_3f_6^3f_8f_{24}^3},\frac{f_3^3f_4^2f_{12}^9}{f_1f_6^5f_8f_{24}^3},\frac{f_1f_4^3f_6^4f_{12}^6}{f_2^3f_3^3f_8f_{24}^3}\right\}, \tag{3.5.7}$$

$$\left\{\frac{f_3^3f_4^2f_{24}}{f_1f_6f_8f_{12}},\frac{f_1f_4^3f_6^8f_{24}}{f_2^3f_3^3f_8f_{12}^4},\frac{f_1^3f_4^2f_6f_{24}}{f_2^2f_3f_8f_{12}},\frac{f_2^7f_3f_{24}}{f_1^3f_4f_6^2f_8}\right\}, \tag{3.5.8}$$

$$\left\{\frac{f_1^2f_4^2f_{12}^3}{f_2f_6f_{24}},\frac{f_2^5f_{12}^3}{f_1^2f_6f_{24}},\frac{f_1^2f_4^2f_6}{f_2},\frac{f_2^5f_6}{f_1^2}\right\}, \tag{3.5.9}$$

$$\left\{\frac{f_2^2f_3f_{12}^{14}}{f_1f_4f_6^6f_{24}^5},\frac{f_1f_{12}^{13}}{f_2f_3f_6^3f_{24}^5},\frac{f_2^2f_3f_6^4}{f_1f_4f_{12}},\frac{f_1f_6^7}{f_2f_3f_{12}^2}\right\}, \tag{3.5.10}$$

(3.5.11) $$\left\{\frac{f_1^2f_4^2f_{12}^{13}}{f_2f_6^5f_{24}^5},\frac{f_2^5f_{12}^{13}}{f_1^2f_6^5f_{24}^5},\frac{f_1^2f_4^2f_6^5}{f_2f_{12}^2},\frac{f_2^5f_6^5}{f_1^2f_{12}^2}\right\},$$

(3.5.12) $$\left\{\frac{f_2^2f_3f_{12}^4}{f_1f_4f_6^2f_{24}},\frac{f_1f_6f_{12}^3}{f_2f_3f_{24}},\frac{f_2^2f_3f_{12}}{f_1f_4},\frac{f_1f_6^3}{f_2f_3}\right\}.$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. Here also proofs of the various statements follow from what has been shown in Section 2:
- (3.5.1) follows from (2.3.64) with $C(q^4) = f_4^2$;
- (3.5.2) follows from (2.3.63) with $C(q^4) = f_4f_8$;
- (3.5.3) follows from (2.3.72) with $C(q^4) = f_4^7f_{24}/(f_8^2f_{12}^2)$;
- (3.5.4) follows from (2.3.73) with $C(q^4) = f_4/f_8$;
- (3.5.5) follows from (2.3.71) with $C(q^4) = 1$;
- (3.5.6) follows from (2.3.65) with $C(q^4) = f_4^4f_{24}/(f_8^2f_{12})$;
- (3.5.7) follows from (2.1.20) with $C(q^2) = f_2^7f_{12}^{10}/(f_4f_6^6f_8f_{24}^3)$;
- (3.5.8) follows from (2.1.20) with $C(q^2) = f_2^7f_{24}/(f_4f_6^2f_8)$;
- (3.5.9) follows from (2.3.76) with $C(q^4) = f_4^2f_{12}^3/f_{24}$;
- (3.5.10) follows from (2.3.74) with $C(q^4) = 1/f_{12}^2$;
- (3.5.11) follows from (2.3.75) with $C(q^4) = f_4^2f_{12}^{13}/f_{24}^5$;
- (3.5.12) follows from (2.3.77) with $C(q^4) = f_{12}^3/f_{24}$. □

Remark: Collections (3.5.1)-(3.5.5) are in group I of Table 6 / Figure 5, while (3.5.9) - (3.5.11) together make up the twelve eta quotients in group VI. Collections (3.5.6), (3.5.7), (3.5.8) and (3.5.12) comprise respectively, groups II, III, IV and VII. Thus the last theorem has completed the task of showing identical vanishing of coefficients within all of these latter groups.

Inclusion results for groups of eta quotients in Table 6 and Figure 5 that are derivable by dissection methods are contained in the next theorem.

Theorem 3.10. *Consider any one of the following pairs of collections of eta quotients* (3.5.13) - (3.5.16)*:*

(3.5.13) $$\left\{\frac{f_2^{11}f_8^2}{f_1^4f_4^5},\frac{f_1^4f_8^2}{f_2f_4}\right\}, \qquad \left\{f_2^3f_8,\frac{f_4^9}{f_2^3f_8^2}\right\},$$

(3.5.14) $$\left\{\frac{f_2^7f_3f_{12}^{10}}{f_1^3f_4f_6^6f_8f_{24}^3},\frac{f_1^3f_4^2f_{12}^9}{f_2^2f_3f_6^3f_8f_{24}^3},\frac{f_3^3f_4^2f_{12}^9}{f_1f_6^5f_8f_{24}^3},\frac{f_1f_3^3f_4^4f_6^6f_{12}}{f_2^3f_3^3f_8f_{24}^3}\right\}, \qquad \left\{\frac{f_4^5f_{12}^8}{f_2^2f_6^3f_8f_{24}^3},\frac{f_2^2f_6^3f_8}{f_4f_{12}}\right\},$$

(3.5.15) $$\left\{\frac{f_3^3f_4^2f_{24}}{f_1f_6f_8f_{12}},\frac{f_1f_4^3f_6^8f_{24}}{f_2^3f_3^3f_8f_{12}^4},\frac{f_1^3f_4^2f_6f_{24}}{f_2^2f_3f_8f_{12}},\frac{f_2^7f_3f_{24}}{f_1^3f_4f_6^2f_8}\right\}, \qquad \left\{\frac{f_4^5f_6f_{24}}{f_2^2f_8f_{12}^2},\frac{f_2^2f_8f_{12}}{f_4f_6}\right\},$$

(3.5.16) $$\left\{\frac{f_1f_3f_4^2f_{24}}{f_2f_8f_{12}},\frac{f_2^2f_4f_6^3f_{24}}{f_1f_3f_8f_{12}^2}\right\}, \qquad \left\{\frac{f_8f_{12}^3}{f_6f_{24}},f_6f_8\right\}.$$

Let $F(q)$ be any eta quotient in the collection on the left, and let $G(q)$ be any eta quotient in the corresponding collection on the right. Then

(3.5.17) $$F_{(0)} \subsetneq G_{(0)}.$$

Proof. As before, and for similar reasons, it is enough in each case to prove each statement for one choice of eta quotient $F(q)$ and likewise one for $G(q)$. The proofs of the various parts of Theorem 3.10 may be summarized as follows:
- (3.5.13) follows from (2.1.9) with $C(q^2) = f_8^2/(f_2f_4)$;
- (3.5.14) follows from (2.1.5) with $C(q^2) = f_4^2f_{12}^9/(f_2^2f_6^3f_8f_{24}^3)$ and (3.5.7);

- (3.5.15) follows from (2.1.5) with $C(q^2) = f_4^2 f_6 f_{24}/(f_2^2 f_8 f_{12})$ and (3.5.8);
- (3.5.16) follows from (2.1.7) with $C(q^2) = f_4^2 f_{24}/(f_2 f_8 f_{12})$. □

Note that parts (3.5.14), (3.5.15) and (3.5.16), respectively, of Theorem 3.10 complete the proof of showing inclusion between, respectively, group III and group IX; group IV and group X; and group V and group XIII of Table 6/Figure 5.

3.6. **Eta quotients with vanishing coefficient behaviour similar to f_1^{26}.** What the experiments in [12] suggested about eta quotients with vanishing coefficient behaviour similar to f_1^{26} is summarized in Table7 and Figure 6, with the * and † symbols and the arrows having the same meaning as elsewhere in the paper.

Table 7: Eta quotients with vanishing behaviour similar to f_1^{26}

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	12	II	4
III *	4	IV †	2
V †	2	VI †	2
VII †	2	VIII	4
IX	8	X †	2
XI	8	XII †	2
XIII	12	XIV	10
XV †	2	XVI †	2
XVII †	4	XVIII †	6

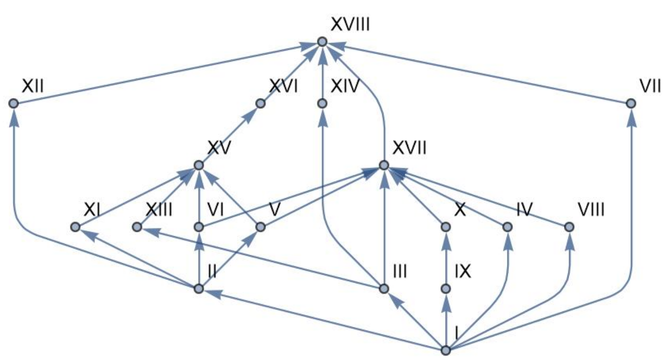

FIGURE 6. The grouping of eta-quotients in Table 7, which have vanishing coefficient behaviour similar to f_1^{26}

This time our dissection methods have less success, in that we are able to prove fewer results about identical vanishing coefficients, and no inclusion results whatsoever.

Theorem 3.11. *Let $F(q)$ and $G(q)$ be any two eta quotients from any one of the following lists.*

(3.6.1)
$$\left\{\frac{f_2^2 f_4^6}{f_1^2}, \frac{f_1^2 f_4^8}{f_2^4}, \frac{f_2^8 f_8^3}{f_1^2 f_4^3}, \frac{f_1^2 f_2^2 f_8^3}{f_4}\right\},$$

(3.6.2)
$$\left\{\frac{f_1^2 f_8^4}{f_4^2}, \frac{f_2^6 f_8^4}{f_1^2 f_4^4}, \frac{f_1^2 f_4 f_8^3}{f_2^2}, \frac{f_2^4 f_8^3}{f_1^2 f_4}\right\},$$

$$\left\{\frac{f_2^5 f_3^2 f_{12}^2}{f_4^2 f_6}, \frac{f_2^5 f_6^5}{f_3^2 f_4^2}, \frac{f_3^2 f_4^{13} f_{12}^2}{f_2^5 f_6 f_8^5}, \frac{f_4^{13} f_6^5}{f_2^5 f_3^2 f_8^5}\right\}, \tag{3.6.3}$$

$$\left\{\frac{f_2 f_3^2 f_{12}^2}{f_6}, \frac{f_2 f_6^5}{f_3^2}, \frac{f_3^2 f_4^3 f_{12}^2}{f_2 f_6 f_8}, \frac{f_4^3 f_6^5}{f_2 f_3^2 f_8}\right\}. \tag{3.6.4}$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. Once again, proofs of the various statements follow from what has been shown in Section 2:
- (3.6.1) follows from (2.3.64) with $C(q^4) = f_8^3/f_4$;
- (3.6.2) follows from (2.3.63) with $C(q^4) = f_8^4/f_4^2$;
- (3.6.3) follows from (2.3.69) with $C(q^4) = f_{12}^2/f_4^2$;
- (3.6.4) follows from (2.3.70) with $C(q^4) = f_{12}^2$. □

Note that (3.6.1) shows identical vanishing of coefficients for four eta quotients in group I, and (3.6.2) completes the proof of showing identical vanishing of coefficients for the four eta quotients in group III. Together, (3.6.3) and (3.6.4) show identical vanishing of coefficients in two distinct collections of four eta quotients each in group IX.

3.7. **Eta quotients with vanishing coefficient behaviour similar to $f_1^3 f_2^3$.** We next summarize what the experiments in [12] suggested about eta quotients with vanishing coefficient behaviour similar to $f_1^3 f_2^3$, as indicated in Table 8 and Figure 7, with the * and † symbols and the arrows having the same meaning as previously.

Table 8: Eta quotients with vanishing behaviour similar to $f_1^3 f_2^3$

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	40	II *	6
III †	2	IV †	2
V †	2	VI †	2
VII †	2	VIII	8
IX	14	X †	2
XI *	4	XII *	4
XIII	10	XIV †	2
XV †	2	XVI †	2
XVII †	6	XVIII †	6

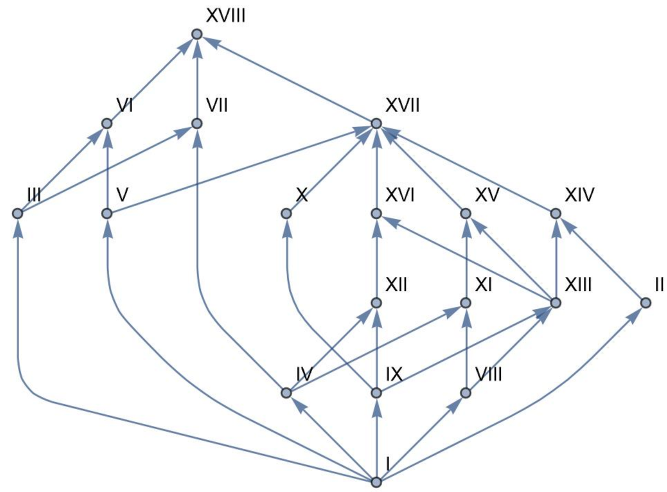

FIGURE 7. The grouping of eta-quotients in Table 8, which have vanishing coefficient behaviour similar to $f_1^3 f_2^3$

This time it is 3-dissections that allow us to prove statements about sets of eta quotients with identically vanishing coefficients.

Theorem 3.12. *Let $F(q)$ and $G(q)$ be any two eta quotients from any one of the following lists.*

$$\left\{\frac{f_1 f_2 f_{12}}{f_6}, \frac{f_2^4 f_{12}}{f_1 f_4 f_6}, \frac{f_1^2 f_4^2 f_6}{f_2^2 f_3}, \frac{f_2^4 f_3 f_{12}}{f_1^2 f_6^2}, \frac{f_1^4 f_{12}}{f_2^2 f_3}, \frac{f_2^{10} f_3 f_{12}^2}{f_1^4 f_4^4 f_6^3}\right\}, \tag{3.7.1}$$

$$\left\{\frac{f_1 f_4 f_6^5}{f_2 f_{12}^2}, \frac{f_2^2 f_6^5}{f_1 f_{12}^2}, \frac{f_1^2 f_6^6}{f_2 f_3 f_{12}^2}, \frac{f_2^5 f_3 f_6^3}{f_1^2 f_4^2 f_{12}}\right\}, \tag{3.7.2}$$

$$\left\{\frac{f_2^2 f_3^5 f_{12}}{f_4 f_6^3}, \frac{f_2^2 f_6^{12}}{f_3^5 f_4 f_{12}^4}, \frac{f_4^2 f_6^{13}}{f_2 f_3^5 f_{12}^5}, \frac{f_3^5 f_4^2}{f_2 f_6^2}\right\}, \tag{3.7.3}$$

$$\left\{\frac{f_1 f_4 f_6}{f_2}, \frac{f_2^2 f_6}{f_1}, \frac{f_1^2 f_6^2}{f_2 f_3}, \frac{f_2^5 f_3 f_{12}}{f_1^2 f_4^2 f_6}\right\}, \tag{3.7.4}$$

$$\left\{\frac{f_2^2 f_3 f_{12}}{f_4 f_6}, \frac{f_2^2 f_6^2}{f_3 f_4}, \frac{f_3 f_4^2}{f_2}, \frac{f_4^2 f_6^3}{f_2 f_3 f_{12}}\right\}. \tag{3.7.5}$$

Then

$$F_{(0)} = G_{(0)}.$$

Proof. Once again proofs of the various statements follow from what has been shown in Section 2:
- (3.7.1) follows from (2.2.14) with $C(q^3) = f_3 f_{12}/f_6^2$;
- (3.7.2) follows from (2.2.10) with $C(q^3) = f_6^5/f_{12}^2$;
- (3.7.3) follows from (2.2.18) with $C(q^3) = f_3^5/f_6^2$;
- (3.7.4) follows from (2.2.10) with $C(q^3) = f_6$;
- (3.7.5) follows from (2.2.18) with $C(q^3) = f_3$. □

Note that (3.7.1), (3.7.4) and (3.7.5) complete the proofs of showing identical vanishing of coefficients in groups II, XI and XII, respectively, while (3.7.2) and (3.7.3) give partial results for groups VIII and IX, respectively.

4. Concluding Remarks

All of the vanishing coefficient results proved in the present paper employed m-dissections, for $m \in \{2,3,4\}$, and all of the m-dissections derived in this paper were derived with the purpose of proving such results. It may be the case that other vanishing coefficient results could be proved using m-dissections for $m > 4$.

We remark in closing that being able to derive m-dissections (some positive integer m) for two eta quotients, eta quotients that experiment suggests have identically vanishing coefficients, may not be enough to prove what experiment suggests. For example, experiment suggests that if

$$B(q) = \frac{f_3^4 f_4 f_{12} f_{18}^3}{f_1 f_6^3 f_9^2 f_{36}}, \qquad C(q) = \frac{f_2^3 f_3^3 f_{12}}{f_1^2 f_4 f_6^2},$$

then $B(q)$ and $C(q)$ have identically vanishing coefficients (and also that each has identically vanishing coefficients with $A(q) = f_1^3 f_2^3$). Further, experiment also suggests that $B(q)$ and $C(q)$ have similar 3-dissections. By writing $f_4/f_1 = (f_1 f_4/f_2)(f_2/f_1^2)$ and employing (2.2.2) and (2.2.5), one can obtain its 3-dissection. Likewise, by writing $f_2^3/(f_1^2 f_4) = (f_2^2/f_4)(f_2/f_1^2)$ and using (2.2.3) (with q replaced with q^2) and (2.2.5), one gets the 3-dissection of $f_2^3/(f_1^2 f_4)$. From these one easily gets the 3-dissections

$$\begin{aligned} B(q) &= \frac{f_3^4 f_4 f_{12} f_{18}^3}{f_1 f_6^3 f_9^2 f_{36}} \\ &= \left(\frac{f_9^2 f_{12}^2 f_{18}^5}{f_3^3 f_6 f_{36}^3} - 4q^3 \frac{f_{12} f_{18}^5}{f_3^2 f_6 f_9} \right) + q \left(\frac{2 f_{12}^2 f_{18}^8}{f_3^2 f_6^2 f_9 f_{36}^3} - \frac{f_6 f_9^5 f_{12}}{f_3^4 f_{18}} \right) + q^2 \left(\frac{4 f_{12}^2 f_{18}^{11}}{f_3 f_6^3 f_9^4 f_{36}^3} - \frac{2 f_9^2 f_{12} f_{18}^2}{f_3^3} \right), \\ C(q) &= \frac{f_2^3 f_3^3 f_{12}}{f_1^2 f_4 f_6^2} \\ &= \left(\frac{f_6^2 f_{12} f_9^6}{f_3^5 f_{18} f_{36}} - 4q^3 \frac{f_6^2 f_{36}^2 f_9^3}{f_3^4 f_{18}} \right) + 2q \left(\frac{f_6 f_{12} f_{18}^2 f_9^3}{f_3^4 f_{36}} - 4q^3 \frac{f_6 f_{18}^2 f_{36}^2}{f_3^3} \right) + q^2 \left(\frac{4 f_{12} f_{18}^5}{f_3^3 f_{36}} - \frac{2 f_6^3 f_9^6 f_{36}^2}{f_3^5 f_{18}^4} \right). \end{aligned}$$

As expected, the corresponding components in the two 3-dissections appear (experimentally) to have identically vanishing coefficients, and indeed it appears (again experimentally) that the two dissections are similar (in the sense defined at (1.0.5) and the remark immediately following) but it is not immediately obvious why that should be the case.

We leave it as a challenge to the reader to prove that $B(q)$ and $C(q)$ have identically vanishing coefficients, and similar 3-dissections.

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