RAMANUJAN AND EXTENSIONS AND CONTRACTIONS OF CONTINUED FRACTIONS

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Abstract. If a continued fraction $K_{n=1}^{\infty} a_n/b_n$ is known to converge but its limit is not easy to determine, it may be easier to use an extension of $K_{n=1}^{\infty} a_n/b_n$ to find the limit. By an extension of $K_{n=1}^{\infty} a_n/b_n$ we mean a continued fraction $K_{n=1}^{\infty} c_n/d_n$ whose odd or even part is $K_{n=1}^{\infty} a_n/b_n$. One can then possibly find the limit in one of three ways:

(i) Prove the extension converges and find its limit;

(ii) Prove the extension converges and find the limit of the other contraction (for example, the odd part, if $K_{n=1}^{\infty} a_n/b_n$ is the even part);

(ii) Find the limit of the other contraction and show that the odd and even parts of the extension tend to the same limit.

We apply these ideas to derive new proofs of certain continued fraction identities of Ramanujan and to prove a generalization of an identity involving the Rogers-Ramanujan continued fraction, which was conjectured by Blecksmith and Brillhart.

1. Introduction

The methods used by the great Indian mathematician, Srinivasa Ramanujan, to obtain many of his fascinating results remain a mystery. In this paper we describe some simple ideas concerning extensions and contractions of continued fractions which may have led Ramanujan to some of the elegant entries concerning continued fractions that he made in his famous notebooks (See [2], Chapter 12).

Suppose we are given a continued fraction $d_0 + K_{n=1}^{\infty} c_n/d_n$ whose limit is sought. If the limit is difficult to compute, it may be easier to work with one of several extensions of the continued fraction which can easily be written down. Suppose, for example, that the even part of an extension gives the original continued fraction. One can try to find the limit in one of three other ways:

(i) Prove the extension converges and find its limit;

(ii) Prove the extension converges and find the limit of the odd part;

(iii) Find the limit of the odd part and show that the odd and even parts of the extension tend to the same limit (by showing that the absolute value of

Date: January 29, 2003.

1991 *Mathematics Subject Classification.* Primary:11A55.

Key words and phrases. Continued Fractions.

the difference between consecutive approximants of the extension tends to 0).

For (i) above, an equivalence transformation of the extended continued fraction may result in its approximants naturally taking on a particulary simple form, so that its limit (and thus of its even part – the original continued fraction) can be found very easily (See, for example, the proofs of Entry 12 and Entry 13).

For (ii) and (iii), it may turn out that the limit of the odd part of the extension can be computed almost trivially. If one can then show that either the extension converges *or* that the even and odd parts of the extension tend to the same limit, then one knows that the limit of the original continued fraction is the same as the limit of the odd part of the extension. The advantage here is that it is usually much easier to show convergence than to determine the actual limit of a continued fraction.

This is our reason for believing that Ramanujan may have used extensions and contractions in the discovery of some of his results – in several entries the odd part of an extension of the continued fraction being considered by Ramanujan can be shown to converge to give Ramanujan's claimed limit almost trivially.

We illustrate the principles involved by giving new proofs of some of Ramanujan's continued fraction identities found in Chapter 12 of the second notebook.

We also use these methods to give a generalization of an identity involving the famous Rogers-Ramanujan continued fraction, first conjectured by Blecksmith and Brillhart [5], and proved by Berndt and Yee in [3].

2. Extensions and Contractions of Continued Fractions

We start with the concepts of extensions and contractions of continued fractions. Before coming to details, we borrow some notation from [14] (page 83).[1] A continued fraction $d_0 + K_{n=1}^{\infty} c_n/d_n$ is said to be a *contraction* of the continued fraction $b_0 + K_{n=1}^{\infty} a_n/b_n$ if its classical approximants $\{g_n\}$ form a subsequence of the classical approximants $\{f_n\}$ of $b_0 + K_{n=1}^{\infty} a_n/b_n$. In this case $b_0 + K_{n=1}^{\infty} a_n/b_n$ is called an *extension* of $d_0 + K_{n=1}^{\infty} c_n/d_n$.

We call $d_0 + K_{n=1}^{\infty} c_n/d_n$ a *canonical contraction* of $b_0 + K_{n=1}^{\infty} a_n/b_n$ if

$$C_k = A_{n_k}, \qquad D_k = B_{n_k} \qquad \text{for } k = 0, 1, 2, 3, \dots,$$

where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0 + K_{n=1}^{\infty} c_n/d_n$ and $b_0 + K_{n=1}^{\infty} a_n/b_n$ respectively.

[1]The authors mention in [14] that this idea also goes back to Seidel [16] and that Lagrange had some special cases already in 1774 [12] and 1776 [11].

Here we use the standard notations

$$\mathop{K}_{n=1}^{N} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots + \cfrac{a_N}{b_N}}}}$$

$$= \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots\frac{a_N}{b_N}.$$

We write A_N/B_N for the above finite continued fraction written as a rational function of the variables $a_1, ..., a_N, b_1, ..., b_N$. By $K_{n=1}^{\infty} a_n/b_n$ we mean the limit of the sequence $\{A_n/B_n\}$ as n goes to infinity, if the limit exists. The ratio A_N/B_N is called the N-th *approximant* of the continued fraction. It is elementary that the A_N (the Nth (canonical) *numerator*) and B_N (the Nth (canonical) *denominator*) satisfy the following recurrence relations:

$$A_N = b_N A_{N-1} + a_N A_{N-2}, \qquad (2.1)$$
$$B_N = b_N B_{N-1} + a_N B_{N-2}.$$

It can also be easily shown that

$$A_N B_{N-1} - A_{N-1} B_N = (-1)^{n-1} \prod_{i=1}^{N} a_i, \qquad (2.2)$$

$$A_{N+1} B_{N-1} - A_{N-1} B_{N+1} = (-1)^{n-1} b_{N+1} \prod_{i=1}^{N} a_i.$$

From [14] (page 83) we have the following theorem:

Theorem 1. *The canonical contraction of* $b_0 + K_{n=1}^{\infty} a_n/b_n$ *with*

$$C_k = A_{2k} \qquad D_k = B_{2k} \qquad \text{for } k = 0, 1, 2, 3, \ldots,$$

exists if and only if $b_{2k} \neq 0 for K = 1, 2, 3, \ldots,$ *and in this case is given by*

$$b_0 + \frac{b_2 a_1}{b_2 b_1 + a_2} - \frac{a_2 a_3 b_4/b_2}{a_4 + b_3 b_4 + a_3 b_4/b_2} - \frac{a_4 a_5 b_6/b_4}{a_6 + b_5 b_6 + a_5 b_6/b_4} + \cdots. \qquad (2.3)$$

The continued fraction (2.3) is called the *even* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$.

We give some simple corollaries to this theorem, which we will use later.

Corollary 1. *The even part of*

$$d_0 + \frac{c_1}{d_1 - c_2 +}\,\frac{c_2}{1 +}\,\frac{-1}{d_2 - c_3 + 1 +}\,\frac{c_3}{1 +}\,\frac{-1}{d_3 - c_4 + 1 +}\,\frac{c_4}{1 +}\cdots \qquad (2.4)$$

is

$$d_0 + \frac{c_1}{d_1 +}\,\frac{c_2}{d_2 +}\,\frac{c_3}{d_3 +}\,\frac{c_4}{d_4 +}\cdots. \qquad (2.5)$$

Proof. In Theorem 1, set $b_0 = d_0$, $a_1 = c_1$, $b_1 = d_1 - c_2$ and, for $k \geq 1$, $a_{2k} = c_{k+1}$, $a_{2k+1} = -1$, $b_{2k} = 1$ and $b_{2k+1} = d_{k+1} - c_{k+2} + 1$. □

Corollary 2. *The even part of*

$$d_0 + \frac{c_1}{d_1 - 1} + \frac{-1}{1} + \frac{c_2}{d_2 - c_2 + 1} + \frac{-1}{1} + \frac{c_3}{d_3 - c_3 + 1} + \frac{-1}{1} + \cdots \tag{2.6}$$

is

$$d_0 + \frac{c_1}{d_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + \frac{c_4}{d_4} + \cdots. \tag{2.7}$$

Proof. In Theorem 1, set $b_0 = d_0$, $a_1 = c_1$, $b_1 = d_1 + 1$ and, for $k \geq 1$, $a_{2k} = -1$, $a_{2k+1} = c_{k+1}$, $b_{2k} = 1$ and $b_{2k+1} = d_{k+1} - c_{k+1} + 1$. □

Corollary 3. *The even part of*

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{1} + \frac{a_3}{0} + \frac{a_4}{1} + \frac{a_5}{0} + \frac{a_6}{1} + \frac{a_7}{0} + \cdots \tag{2.8}$$

is

$$b_0 + \frac{a_1}{b_1 + a_2} - \frac{a_2 a_3}{a_4 + a_3} - \frac{a_4 a_5}{a_6 + a_5} - \frac{a_6 a_7}{a_8 + a_7} - \cdots. \tag{2.9}$$

Proof. In Theorem 1, set $b_{2k} = 1$ and $b_{2k+1} = 0$, for $k \geq 1$. □

From [14] (page 85) we also have:

Theorem 2. *The canonical contraction of* $b_0 + K_{n=1}^{\infty} a_n/b_n$ *with* $C_0 = A_1/B_1$

$$C_k = A_{2k+1} \qquad D_k = B_{2k+1} \qquad \text{for } k = 1, 2, 3, \ldots,$$

exists if and only if $b_{2k+1} \neq 0 for K = 0, 1, 2, 3, \ldots$, *and in this case is given by*

$$\frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3} - \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \frac{a_7 a_8 b_9/b_7}{a_9 + b_8 b_9 + a_8 b_9/b_7} + \cdots. \tag{2.10}$$

The continued fraction (2.10) is called the *odd* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$.
We will also make use of the following corollary to Theorem 2.

Corollary 4. *The odd part of the continued fraction*

$$\frac{c_1}{1} - \frac{c_2}{1} + \frac{c_2}{1} - \frac{c_3}{1} + \frac{c_3}{1} - \frac{c_4}{1} + \frac{c_4}{1} - \cdots \tag{2.11}$$

is

$$c_1 + \frac{c_1 c_2}{1} + \frac{c_2 c_3}{1} + \frac{c_3 c_4}{1} + \cdots. \tag{2.12}$$

Proof. In Theorem 2, set $b_0 = 0$, $a_1 = c_1$, and, for $k \geq 1$, $b_i = 1$, $a_{2i} = -c_i$ and $a_{2i+1} = c_i$. □

We will not explicitly compute the odd parts of the continued fractions at (2.4), (2.6) and (2.8) at this point.

We also give a new extension/contraction proof of Daniel Bernoulli's transformation of a sequence into a continued fraction [4] (see, for example, [10], pp. 11–12).

Proposition 1. *Let $\{K_0, K_1, K_2, \ldots\}$ be a sequence of complex numbers such that $K_i \neq K_{i-1}$, for $i = 1, 2, \ldots$.*

Then $\{K_0, K_1, K_2, \ldots\}$ is the sequence of approximants of the continued fraction

$$K_0 + \frac{K_1 - K_0}{1} + \frac{K_1 - K_2}{K_2 - K_0} + \frac{(K_1 - K_0)(K_2 - K_3)}{K_3 - K_1} + \cdots + \frac{(K_{n-2} - K_{n-3})(K_{n-1} - K_n)}{K_n - K_{n-2}} + \cdots. \tag{2.13}$$

Proof. We use the fact that

$$\frac{1}{a} + \frac{1}{0} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b} + \frac{1}{c}.$$

Then

$$K_0 + \frac{1}{0} + \frac{1}{K_1 - K_0} + \frac{1}{0} + \frac{1}{K_2 - K_1} + \cdots + \frac{1}{0} + \frac{1}{K_n - K_{n-1}}$$
$$= K_0 + \frac{1}{0} + \frac{1}{\sum_{i=1}^{n} K_i - K_{i-1}} = K_n.$$

On the other hand, by Theorem 1, the even part of the above continued fraction is

$$K_0 + \frac{K_1 - K_0}{1} - \frac{(K_2 - K_1)/(K_1 - K_0)}{1 + (K_2 - K_1)/(K_1 - K_0)} - \frac{(K_3 - K_2)/(K_2 - K_1)}{1 + (K_3 - K_2)/(K_2 - K_1)} - \cdots - \frac{(K_n - K_{n-1})/(K_{n-1} - K_{n-2})}{1 + (K_n - K_{n-1})/(K_{n-1} - K_{n-2})}$$

$$= K_0 + \frac{K_1 - K_0}{1} + \frac{K_1 - K_2}{K_2 - K_0} + \frac{(K_1 - K_0)(K_2 - K_3)}{K_3 - K_1} + \cdots + \frac{(K_{n-2} - K_{n-3})(K_{n-1} - K_n)}{K_n - K_{n-2}}.$$

□

If we let $K_n = \sum_{i=0}^{n} a_i$, we of course get Euler's transformation of a series into a continued fraction:

$$\sum_{i=0}^{n} a_i = a_0 + \frac{a_1}{1} + \frac{-a_2}{a_2 + a_1} + \frac{-a_1 a_3}{a_3 + a_2} + \cdots \frac{-a_{n-2} a_n}{a_n + a_{n-1}}. \tag{2.14}$$

3. Some Continued Fractions from Chapter 12 of Ramanujan's Second Notebook

In each of the following example, $b_0 + K_{n=1}^{\infty} a_n/b_n$ will mean the extended continued fraction under consideration and $\{A_n\}$ and $\{B_n\}$ will denote its sequences numerators and denominators respectively.

We will frequently make use of the following important theorem of Worpitzky (see [14], pp. 35–36).

Theorem 3. *(Worpitzky) Let the continued fraction* $K_{n=1}^{\infty} a_n/1$ *be such that* $|a_n| \leq 1/4$ *for* $n \geq 1$. *Then*$K_{n=1}^{\infty} a_n/1$ *converges. All approximants of the continued fraction lie in the disc* $|w| < 1/2$ *and the value of the continued fraction is in the disk* $|w| \leq 1/2$.

We now illustrate the methods involved by giving new proofs of several continued fraction identities due to Ramanujan.

Entry 7. ([2], page 112) *If* x *is not a negative integer, then*

$$1 = \frac{x+1}{x} \; + \; \frac{x+2}{x+1} \; + \; \frac{x+3}{x+2} \; + \cdots. \tag{3.1}$$

We prove a generalization of Entry 7.

Entry 7a. *Let* $\{y_i\}_{i=1}^{\infty}$ *be any sequence of complex numbers such that*

$$(i)\, y_i \neq -1, \; i = 1, 2, 3, \ldots,$$

$$(ii)\; \lim_{n\to\infty} \prod_{i=1}^{n} |1 + y_i| = \infty,$$

$$(iii) \left| \frac{y_i + 1}{y_{i-1}\, y_i} \right| \leq \frac{1}{4}, \; \textit{for all } i \geq N_0, \text{ some } N_0.$$

Then

$$1 = \frac{y_1 + 1}{y_1} \; + \; \frac{y_2 + 1}{y_2} \; + \; \frac{y_3 + 1}{y_3} \; + \cdots \tag{3.2}$$

Proof. It is sufficient to assume $N_0 = 2$ (if not, one can prove the result for the tail that begins with y_{N_0} in the numerator and the continued fraction will then collapse from the bottom up to give the result).

After a similarity transformation, the left side of Equation 3.2 becomes

$$\frac{(y_1+1)/y_1}{1} \; + \; \frac{(y_2+1)/(y_1 y_2)}{1} \; + \; \frac{(y_3+1)/(y_2 y_3)}{1} \; + \cdots$$

and Worpitzky's theorem gives that this continued fraction, and thus the left side of (3.2) converges.

From Corollary 2 it can be see that the right side of Equation 3.2 is the even part of

$$\frac{y_1+1}{y_1+1} + \frac{-1}{1} + \frac{y_2+1}{0} + \frac{-1}{1} + \frac{y_3+1}{0} + \frac{-1}{1} + \cdots. \tag{3.3}$$

Thus the even part of (3.3) converges and, from Equation 2.2,

$$0 = \lim_{i\to\infty} \left| \frac{A_{2i+2}}{B_{2i+2}} - \frac{A_{2i}}{B_{2i}} \right| = \lim_{i\to\infty} \left| \frac{A_{2i+2}B_{2i} - A_{2i}B_{2i+2}}{B_{2i+2}B_{2i}} \right| = \lim_{i\to\infty} \frac{\prod_{j=1}^{i+1} |y_j + 1|}{|B_{2i}B_{2i+2}|}$$

Condition (ii) above then gives that

$$\lim_{i\to\infty} |B_{2i}B_{2i+2}| = \infty. \tag{3.4}$$

From the recurrence relations at (2.1) and the fact that $b_{2i+1} = 0$, for $i = 1, 2, \ldots$, one has that

$$A_{2i+1} = (y_{i+1} + 1)A_{2i-1} = \cdots = \prod_{j=1}^{i+1} (y_j + 1).$$

Similarly, $B_{2i+1} = \prod_{j=1}^{i+1} (y_j + 1)$ and so each odd-numbered approximant is identically 1. From Equation 2.2 above it also follows that

$$|A_{2i+1}B_{2i} - A_{2i}B_{2i+1}| = \prod_{j=1}^{i+1} |y_j + 1| = |B_{2i+1}|.$$

Thus

$$\left|1 - \frac{A_{2i}}{B_{2i}}\right| = \left|\frac{A_{2i+1}}{B_{2i+1}} - \frac{A_{2i}}{B_{2i}}\right| = \left|\frac{A_{2i+1}B_{2i} - A_{2i}B_{2i+1}}{B_{2i}B_{2i+1}}\right| = \frac{1}{|B_{2i}|}. \tag{3.5}$$

Since the limit of the left side exists, it follows that $\lim_{i\to\infty} |B_{2i}|$ exists and Equation 3.4 gives that this limit is ∞. Thus $\lim_{i\to\infty} A_{2i}/B_{2i} = 1$ and the result follows. □

For the next example, we will show that the extended continued fraction converges, so that the even and odd parts have the same limit. We will give two different proofs to better illustrate the methods. One will use the following theorem of Lange [13] (see [9], page 124):

Theorem 4. *The continued fraction $K(c_n^2/1)$ converges to a finite value provided that*

$$|c_{2n-1} \pm i\alpha| \leq \rho, \quad |c_{2n} \pm i(1+\alpha)| \geq \rho, \quad n = 1, 2, 3, \ldots, \tag{3.6}$$

where α is a complex number and α and ρ satisfy the inequality

$$|\alpha| < \rho < |\alpha + 1|. \tag{3.7}$$

The convergence is uniform with respect to the regions defined by (3.6).

(We have changed the notation in the above theorem slightly to avoid conflict with existing notation.)

The other proof will use the following theorem, due to Wall [18] (see [9], page 127):

Theorem 5. *Let $\{f_n\}$ be the sequence of approximants of a continued fraction $K(a_n/1)$. Assume that there exist positive numbers M, L, n_0 and a subsequence $\{m_k\}$ of the positive integers such that*

$$|f_n| < M, \qquad n = n_0, n_0 + 1, n_0 + 2, \ldots, \tag{3.8}$$

and

$$|a_{m_k}| < L, \qquad k = 1, 2, 3, \ldots. \tag{3.9}$$

Further assume that the odd (even) part of $K(a_n/1)$ converges to a finite value v. Then there exists a subsequence of the even (odd) part which converges to v.

Remark: An obvious implication of this theorem is that if, in addition, the odd and even parts both converge, then they converge to the same limit and the continued fraction converges to this limit.

Entry 9. ([2], page 114) *Let a and x be complex numbers such that either $x \neq -k\,a$ for $k \in \{1, 2, \ldots\}$ and $a \neq 0$, or that $a = 0$ and $|x| > 1$. Then*

$$\frac{x+a+1}{x+1} = \frac{x+a}{x-1} + \frac{x+2a}{x+a-1} + \frac{x+3a}{x+2a-1} + \cdots. \tag{3.10}$$

We will not consider the case $a = 0$, since the right side is periodic for $a = 0$ and the result follows from a general theorem for periodic continued fractions.

Neither will we consider the case $x = -1$, since this case ("$\infty = \infty$") follows from the case $x \neq -1$ by considering the tail beginning with $-1+3a$ in the numerator.

FIRST PROOF. We will prove Entry 9 for $a \notin (-\infty, 0)$. From Corollary 2, the right side of Equation 3.10 is the even part of

$$\frac{x+a}{x} + \frac{-1}{1} + \frac{x+2a}{-a} + \frac{-1}{1} + \frac{x+3a}{-a} + \frac{-1}{1} + \frac{x+4a}{-a} + \cdots \tag{3.11}$$

From Theorem 2, the odd part of this latter continued fraction is

$$\begin{aligned}
&\frac{x+a}{x} + \frac{(x+a)(-a)/x}{x(x+a)+a} + \frac{(x+2a)x}{(x+2a)-1} \\
&\quad + \frac{(x+3a)}{(x+3a)-1} + \frac{(x+4a)}{(x+4a)-1} + \cdots \\
&= \frac{x+a}{x} + \frac{(x+a)(-a)/x}{x(x+a)+a+x} \\
&= \frac{x+a+1}{x+1}
\end{aligned}$$

The second last equality follows from Entry 7a applied to the tail of the continued fraction. Thus the odd part of the extension converges to the left side of Equation 3.10. Our first proof that the extension itself converges uses Theorem 4. The continued fraction at (3.11) is equivalent to the following continued fraction:

$$\begin{aligned}
&\frac{(x+a)/x}{1} + \frac{-1/x}{1} + \frac{-x/a-2}{1} + \frac{1/a}{1} + \\
&\qquad \frac{-x/a-3}{1} + \frac{1/a}{1} + \frac{-x/a-4}{1} + \cdots
\end{aligned} \tag{3.12}$$

We consider a tail of this continued fraction

$$\frac{1/a}{1} + \frac{-x/a-m}{1} + \frac{1/a}{1} + \frac{-x/a-m-1}{1} + \frac{1/a}{1} + \frac{-x/a-m-2}{1} + \cdots, \tag{3.13}$$

where m will depend on a and x and will be determined later. If the tail converges, then the continued fraction converges and its limit is $(x+a+1)/(x+1)$, since the odd part converges to this limit. Denote the continued fraction at (3.13) by $K_{k=1}^{\infty} c_k^2/1$, so that

$$c_{2k-1}^2 = \frac{1}{a}, \qquad c_{2k}^2 = -\frac{x}{a} - m - k + 1. \tag{3.14}$$

Note that if α and ρ can be found such that inequality (3.7) and the first inequality in (3.6) can be satisfied, then the second inequality in (3.6) will be satisfied automatically for all k, provided m is chosen large enough. Let $\sqrt{1/a} = c + i\,d$, where $c > 0$ (since $a \notin (-\infty, 0)$). Set

$$\alpha = \frac{c^2+d^2}{2}\left(1 + i\frac{d}{c}\right), \qquad \rho = \sqrt{\frac{c^2+d^2}{4c^2}((c^2+d^2)^2+4c^2)}. \tag{3.15}$$

Then $|\alpha| < \rho < |\alpha+1| = \sqrt{\rho^2+1}$ and $|c_{2k-1} + i\alpha| = |c_{2k-1} - i\alpha| = \rho$. Provided m is chosen large enough so that $|-x/a-m-k+1 \pm i(1+\alpha)| \geq \rho$, for $k = 1, 2, \ldots$, the conditions of Theorem 4 are satisfied, the tail converges to a finite value and the extended continued fraction converges and Entry 9 follows for $a \notin (-\infty, 0)$.

□

SECOND PROOF. We will apply Theorem 5 to the continued fraction at (3.12). Without loss of generality, we can assume that

$$\left|\frac{x+ja}{(x+(j-2)a-1)(x+(j-1)a-1)}\right| \leq \frac{1}{4}$$

holds for $j \geq 2$, since this holds for all j sufficiently large and if Entry 9 holds for a tail of the continued fraction, ie.,

$$\frac{x+na+1}{x+(n-1)a+1} = \frac{x+na}{x+(n-1)a-1} + \frac{x+(n+1)a}{x+na-1} + \frac{x+(n+2)a}{x+(n+1)a-1} + \cdots, \tag{3.16}$$

for some integer n, then Entry 9 is proved, since the continued fraction will then collapse from the bottom up to give the result. Thus Worpitzky's Theorem gives that the even part of 3.11, and thus the even part of 3.12, converges to a finite value and we already know that the odd part converges to the left side of Equation 3.10. Thus, since the odd and even parts both converge to finite values, there exist an n_0 and an M such that Equation 3.8 is satisfied. Further, from (3.12), it is clear that in Equation 3.9, we can take $\{m_k\}$ to be the even integers and L to be $\max\{|1/a|+1, |1/x|+1\}$. (The case $x = 0$ is not a problem since we can consider the tail of the continued fraction beginning with the third partial numerator.) The conditions of Theorem 5 are satisfied and, by the remark following it, the continued fraction at (3.11) converges and Entry 9 follows.

□

Entry 10 ([2], page 116) *If n is a positive integer, then*

$$n = \frac{1}{1-n} + \frac{2}{2-n} + \frac{3}{3-n} + \cdots + \frac{n}{0} + \frac{n+1}{1} + \frac{n+2}{2} + \cdots. \tag{3.17}$$

Proof. The proof is by induction on n. If $n = 1$, the left side of (3.17) is

$$n = \frac{1}{1-1} + \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots = \frac{1}{1-1+1} = 1,$$

by Entry 7. Suppose Entry 10 is true for $n = 1, 2, \ldots, m-1$. From Corollary 2,

$$\frac{1}{1-m} + \frac{2}{2-m} + \frac{3}{3-m} + \cdots + \frac{m}{0} + \frac{m+1}{1} + \frac{m+2}{2} + \cdots \tag{3.18}$$

is the even part of

$$\frac{1}{2-m} + \frac{-1}{1} + \frac{2}{1-m} + \frac{-1}{1} + \frac{3}{1-m} + \cdots. \tag{3.19}$$

From Theorem 2, the odd part of this latter continued fraction is

(3.20)

$$\begin{aligned}
&\frac{1}{2-m} + \frac{(1-m)/(2-m)}{(2-m)(3-m)-(1-m)} + \frac{2(2-m)}{(4-m)-1} \\
&\quad + \frac{3}{(5-m)-1} + \frac{4}{(6-m)-1} + \cdots \\
&= \frac{1}{2-m} + \frac{(1-m)/(2-m)}{(2-m)(3-m)-(1-m)} + \frac{(2-m)2}{2-(m-1)} \\
&\quad + \frac{3}{3-(m-1)} + \frac{4}{4-(m-1)} + \cdots \\
&= \frac{1}{2-m} + \frac{(1-m)/(2-m)}{(2-m)(3-m)-(1-m)+(2-m)\left(\frac{1}{m-1}-(2-m)\right)} \\
&= m.
\end{aligned}$$

The next to last step comes from applying the induction step to the continued fraction in (3.17), when $n = m-1$.

Next, we will use Theorem 4 to show that a tail of the continued fraction at (3.19) converges and thus that the continued fraction itself converges to

m, since the odd part equals m. This continued fraction is equivalent to

(3.21) $$\frac{1/(2-m)}{1} + \frac{1/(m-2)}{1} + \frac{2/(1-m)}{1} + \frac{1/(m-1)}{1} + \frac{3/(1-m)}{1} + \frac{1/(m-1)}{1} + \cdots.$$

We consider a tail of this last continued fraction:

(3.22) $$\frac{1/(m-1)}{1} + \frac{N/(1-m)}{1} + \frac{1/(m-1)}{1} + \frac{(N+1)/(1-m)}{1} + \cdots,$$

where N depends on m and will be chosen later. With the notation of Theorem 4, let this continued fraction be denoted $K_{k=1}^{\infty}(c_k^2/1)$. Since $m \geq 2$, $c_{2k-1} = \sqrt{1/(m-1)}$ is real and we chose the positive square root so that $c_{2k-1} =: c > 0$. Let $c_{2k} = +i\sqrt{(N+k-1)/(m-1)}$. We chose $\alpha = c^2/2$ and $\rho = \sqrt{c^4/4 + c^2}$. Then

$$|\alpha| < \rho < |\alpha + 1| = \sqrt{\rho^2 + 1}.$$

Further,

$$|c_{2k-1} + i\,\alpha| = |c_{2k-1} - i\,\alpha| = \rho.$$

For N sufficiently large, $|c_{2k} \pm i(1+\alpha)| \geq \rho$. The conditions of Theorem 4 are satisfied, a tail of (3.21) converges and, by the remark following the statement of this theorem, (3.21) itself converges and Entry 10 follows. □

Entry 12. ([2], page 118) *If $a \neq 0$ and $x \neq -k\,a$, where k is a positive integer,*

(3.23)
$$1 = \frac{x+a}{a} + \frac{(x+a)^2 - a^2}{a} + \frac{(x+2a)^2 - a^2}{a} + \frac{(x+3a)^2 - a^2}{a} + \cdots.$$

Proof. The left side of Equation 3.23 is

(3.24) $$\frac{x+a}{a} + \frac{x(x+2a)}{a} + \frac{(x+a)(x+3a)}{a} + \cdots + \frac{(x+(k-1)a)(x+(k+1)a)}{a} + \cdots,$$

which, by Corollary 3, is the even part of

(3.25) $$\frac{x+a}{x+a} + \frac{-x}{1} + \frac{x+2a}{0} + \frac{-(x+a)}{1} + \frac{x+3a}{0} + \frac{-(x+2a)}{1} + \cdots.$$

This continued fraction is equivalent to the following continued fraction:

(3.26) $$\frac{1}{1} + \frac{1}{-1-a/x} + \frac{1}{0} + \frac{1}{1+2a/x} + \frac{1}{0} + \frac{1}{-1-3a/x} + \frac{1}{0} + \frac{1}{1+4a/x} + \frac{1}{0} + \frac{1}{-1-5a/x} + \frac{1}{0} + \frac{1}{1+6a/x} + \cdots$$

By the similar reasoning to that used in the proof of Entry 7a, each odd-indexed approximant of (3.26) is identically 1. To calculate the even-indexed approximants, we make repeated use the identity

$$\frac{1}{a}+\frac{1}{0}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b}+\frac{1}{c} \tag{3.27}$$

to simplify the approximant. One easily checks that

$$\begin{aligned}&\frac{1}{1}+\frac{1}{-1-a/x}+\frac{1}{0}+\frac{1}{1+2a/x}+\frac{1}{0}+\cdots+\frac{1}{0}+\frac{1}{1+2ka/x}\\&=\frac{1}{1}+\frac{1}{(-1-a/x)+(1+2a/x)+(-1-3a/x)+\cdots+(1+2ka/x)}\\&=\frac{1}{1}+\frac{1}{ka/x}.\end{aligned}$$

Similarly, it is easy to check that

$$\begin{aligned}&\frac{1}{1}+\frac{1}{-1-a/x}+\frac{1}{0}+\frac{1}{1+2a/x}+\frac{1}{0}+\cdots+\frac{1}{0}+\frac{1}{-1-(2k-1)a/x}\\&=\frac{1}{1}+\frac{1}{(-1-a/x)+(1+2a/x)+(-1-3a/x)+\cdots+(-1-(2k-1)a/x)}\\&=\frac{1}{1}+\frac{1}{-1-ka/x}.\end{aligned}$$

Upon letting $k\to\infty$, one has that the even-indexed tend to 1 also and Entry 12 is proved. □

Before coming to Entry 13, we state the following theorem of Hill [6] (see: [1], page 63)

Theorem 6. *Let s_n denote the nth partial sum of ${}_2F_1(a,b;c;1)$. For Re$(c-a-b)<0$,*

$$s_n \sim \frac{\Gamma(c)n^{a+b-c}}{\Gamma(a)\Gamma(b)}, \tag{3.28}$$

and for $c=a+b$,

$$s_n \sim \frac{\Gamma(c)\log n}{\Gamma(a)\Gamma(b)}. \tag{3.29}$$

Here

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,$$

where $(d)_n=d(d+1)\dots(d+n-1)$ for $n>0$ and $(d)_0=1$.

Entry 13. ([2], page 119) *Let a, b and d be complex numbers such that either $d\neq 0$, $b\neq -kd$, where k is a non-negative integer, and* Re$((a-$

b)/*d*) < 0, [2] *or* $d \neq 0$ *and* $a = b$, *or* $d = 0$ *and* $|a| < |b|$. *Then*

$$a = \frac{ab}{a+b+d\,-}\,\frac{(a+d)(b+d)}{a+b+3d}\;-\;\frac{(a+2d)(b+2d)}{a+b+5d}\;-\cdots \tag{3.30}$$

Proof. We make the further assumption $a + kd \neq 0$, for k a non-negative integer. By Corollary 3, the right side of Equation 3.30 is the even part of the continued fraction

$$\frac{ab}{b}\,+\,\frac{a+d}{1}\,+\,\frac{b+d}{0}\,+\,\frac{a+2d}{1}\,+\,\frac{b+2d}{0}\,+\cdots\frac{a+kd}{1}\,+\,\frac{b+kd}{0}\,+\cdots. \tag{3.31}$$

This latter continued fraction is equivalent to

$$\frac{1}{1/a}\,+\,\frac{1}{\frac{ab}{a+d}}\,+\,\frac{1}{0}\,+\,\frac{1}{\frac{ab(b+d)}{(a+d)(a+2d)}}\,+\,\frac{1}{0}\,+\,\frac{1}{\frac{ab(b+d)(b+2d)}{(a+d)(a+2d)(a+3d)}}\,+\,\frac{1}{0}\,+\,\frac{1}{\frac{ab(b+d)(b+2d)(b+3d)}{(a+d)(a+2d)(a+3d)(a+4d)}}\,+\,\frac{1}{0}\,+\cdots. \tag{3.32}$$

By similar reasoning to that used in Example7a, each odd-indexed approximant is identically equal to a. We now consider the even-indexed approximants, treating each of the three cases in the statement of Entry 13 in turn. Note that, since $d \neq 0$ (as in the first case),

$$\frac{ab(b+d)\ldots(b+(k-1)d)}{(a+d)(a+2d)\ldots(a+kd)} = a\frac{(b/d)_k}{(a/d+1)_k}. \tag{3.33}$$

Let $\{f_k\}$ denote the sequence of approximants for the continued fraction at (3.32). By using the same collapsing technique as was used in the proof of Example 12,

$$f_{2k} = \frac{1}{1/a}\,+\,\frac{1}{a\sum_{i=1}^{k}\frac{(b/d)_i}{(a/d+1)_i}} = \frac{1}{1/a}\,+\,\frac{1}{-a+a\sum_{i=0}^{k}\frac{(b/d)_i}{(a/d+1)_i}} \tag{3.34}$$

[2] Entry 13, as written in [2], page 119, reads "Re($(a-b)/d$) > 0". This inequality should be reversed. The following example is an indication of this fact (a complete proof that the reversed inequality is the correct one is found in the proof of Entry 13 above): Let $a = 2$ and $b = d = 1$, so that $\mathrm{Re}((a-b)/d) > 0$. However the left side of (3.30) is

$$\frac{2.1}{4}\,+\,\frac{-3.2}{6}\,+\,\frac{-4.3}{8}\,+\,\frac{-5.4}{10}\,+\,\frac{-6.5}{12}\,+\cdots$$
$$=\frac{1}{2}\,+\,\frac{-1}{2}\,+\,\frac{-1}{2}\,+\,\frac{-1}{2}\,+\,\frac{-1}{2}\,+\cdots.$$

This continued fraction has the sequence of approximants $\{n/(n+1)\}$ and therefore converges to 1 (=b) and not 2 (=a).

We note that $\sum_{i=0}^{k} \frac{(b/d)_i}{(a/d+1)_i}$ is the kth partial sum of ${}_2F_1(1, b/d; a/d+1; 1)$. Since $\text{Re}(a/d + 1 - b/d - 1) = \text{Re}((a-b)/d) < 0$, we have by Theorem 6 that

$$\sum_{i=0}^{k} \frac{(b/d)_i}{(a/d+1)_i} \sim \frac{\Gamma(a/d+1)k^{1+b/d-a/d-1}}{\Gamma(1)\Gamma(b/d)} = \frac{\Gamma(a/d+1)k^{(b-a)/d}}{\Gamma(1)\Gamma(b/d)}. \tag{3.35}$$

Thus, since $\text{Re}((b-a)/d) > 0$,

$$\lim_{k\to\infty} \sum_{i=0}^{k} \frac{(b/d)_i}{(a/d+1)_i} = \infty$$

and from Equation 3.34, $\lim_{k\to\infty} f_{2k} = a$.

Next, suppose $d \neq 0$ and $a = b$. After canceling common factors in each denominator and collapsing the continued fraction as before, we have that

$$f_{2k} = \frac{1}{1/a} + \frac{1}{\displaystyle\sum_{i=1}^{k} \frac{a^2}{a+id}}, \tag{3.36}$$

and once again it is clear that $\lim_{k\to\infty} f_{2k} = a$.

Finally, if $d = 0$ and $|a| < |b|$

$$f_{2k} = \frac{1}{1/a} + \frac{1}{a\displaystyle\sum_{i=1}^{k} \left(\frac{b}{a}\right)^i}, \tag{3.37}$$

and once again $\lim_{k\to\infty} f_{2k} = a$. Entry 13 is proved. □

Remark: Interestingly, this proof, deriving from extending the right side of (3.30), coincides at the finish with Jacobsen's proof [8], which uses a theorem, due to her [7] and Waadeland [17], on tails of continued fractions. Both proofs eventually rely on Hill's result from Theorem 6 applied to the same ${}_2F_1$ function.

4. An Extension of the Rogers-Ramanujan Continued Fraction

For $|q| < 1$, let

$$R(q) := 1 + \frac{q}{1} {\atop +} \frac{q^2}{1} {\atop +} \frac{q^3}{1} {\atop +} \cdots, \tag{4.1}$$

the famous Rogers-Ramanujan continued fraction. In [5], J. Brillhart and R. Blecksmith conjectured that

$$R(q) = \frac{1}{1} {\atop -} \frac{q}{1} {\atop +} \frac{q}{1} {\atop -} \frac{q}{1} {\atop +} \frac{q}{1} {\atop -} \frac{q^2}{1} {\atop +} \frac{q^2}{1} {\atop -} \frac{q^2}{1} {\atop +} \frac{q^2}{1} {\atop -} \cdots. \tag{4.2}$$

This conjecture was proved by Berndt and Yee in [3]. We generalize this result as follows. For q, $\alpha \in \mathbb{C}$, let q^α be defined as usual by

$$q^\alpha = e^{\alpha \log q},$$

where $\log q$ is the principal logarithm of q.

Proposition 2. *Let q, $\alpha \in \mathbb{C}$, with $|q| < 1$. Then*

$$(4.3) \quad R(q) = 1 - q^\alpha + \frac{q^\alpha}{1} - \frac{q^{1-\alpha}}{1} + \frac{q^{1-\alpha}}{1} - \frac{q^{1+\alpha}}{1} + \frac{q^{1+\alpha}}{1} - \frac{q^{2-\alpha}}{1} + \frac{q^{2-\alpha}}{1} - \frac{q^{2+\alpha}}{1} + \frac{q^{2+\alpha}}{1} - \cdots.$$

Remark: The conjecture of Blecksmith and Brillhart is the $\alpha = 0$ case of this proposition.

Proof. In Corollary 4, let $c_1 = q^\alpha$ and, for $k \geq 1$, let $c_{2k} = q^{k-\alpha}$ and $c_{2k+1} = q^{k+\alpha}$. This gives that the odd part of

$$\frac{q^\alpha}{1} - \frac{q^{1-\alpha}}{1} + \frac{q^{1-\alpha}}{1} - \frac{q^{1+\alpha}}{1} + \frac{q^{1+\alpha}}{1} - \frac{q^{2-\alpha}}{1} + \frac{q^{2-\alpha}}{1} - \frac{q^{2+\alpha}}{1} + \frac{q^{2+\alpha}}{1} - \cdots$$

is

$$q^\alpha + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots.$$

Since a tail of the left side of (4.3) converges (by Worpitzky's Theorem) and its odd part converges to $R(q)$, this proves the result. □

We also have the following corollary.

Corollary 5. *Let q, $\alpha \in \mathbb{C}$, with $|q| < 1$. Then*

$$(4.4) \quad R(q) = 1 - q^\alpha + \frac{q^\alpha}{1 - q^{1-\alpha}} + \frac{q^{2(1-\alpha)}}{1 + q^{1-\alpha} - q^{1+\alpha}} + \frac{q^{2(1+\alpha)}}{1 + q^{1+\alpha} - q^{2-\alpha}} + \frac{q^{2(2-\alpha)}}{1 + q^{2-\alpha} - q^{2+\alpha}} + \frac{q^{2(2+\alpha)}}{1 + q^{2+\alpha} - q^{3-\alpha}} + \cdots$$

Proof. By Theorem 1, the left side of Equation 4.4 is the even part of the continued fraction at (4.3). □

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