SOME MORE LONG CONTINUED FRACTIONS, I

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Abstract. In this paper we show how to construct several infinite families of polynomials $D(\bar{x}, k)$, such that $\sqrt{D(\bar{x}, k)}$ has a regular continued fraction expansion with arbitrarily long period, the length of this period being controlled by the positive integer parameter k.

We also describe how to quickly compute the fundamental units in the corresponding real quadratic fields.

1. Introduction

Let $D(x_1, x_2, \ldots, x_r, k)$ be a polynomial in $\mathbb{Z}[x_1, x_2, \ldots, x_r]$, where x_1, x_2,$\ldots, x_r$ are integer variables and k is a positive integer parameter which appears as an exponent in the expression of $D(x_1, x_2, \ldots, x_r, k)$. Suppose further that there are positive lower bounds T_1 and T_2 such that, for all integral $x_i \geq T_1$ and all integral $k \geq T_2$, the surd $\sqrt{D(x_1, x_2, \ldots, x_r, k)}$ has a regular continued fraction of the form

$$\sqrt{D(x_1, x_2, \ldots, x_r, k)} = [\, a_0; \overline{a_1, \cdots, a_{n_k}, 2a_0}],$$

where each $a_j := a_j(x_1, \cdots, x_r, k) \in \mathbb{Z}[x_1, \cdots, x_r]$, for $j = 0, 1, \ldots n_k$, and the length of the period, $n_k + 1$, depends only on k (k may also be present in some of the a_j as an exponent). Under these circumstances we say that $D(x_1, x_2, \ldots, x_r, k)$ has a *long continued fraction expansion* and call the expansion $[\, a_0; \overline{a_1, \cdots, a_{n_k}, 2a_0}]$ a *long continued fraction.*

We give the following example due to Madden [7] to illustrate the concept: Let b, n and k be positive integers. Set

$$D := D(b, n, k) := \Big(b\,(1 + 2\,b\,n)^k + n\Big)^2 + 2\,(1 + 2\,b\,n)^k.$$

Then

$$\begin{aligned}\sqrt{D} = \Big[& b(1+2bn)^k + n; \overline{b, 2b(1+2bn)^{k-1}, b(2bn+1), 2b(2bn+1)^{k-2},} \\ & \overline{b(2bn+1)^2, 2b(2bn+1)^{k-3}, \ \ldots, \ b(1+2bn)^{k-1}, 2b,} \\ & \overline{n + b(2bn+1)^k,} \\ & \overline{2b, b(1+2bn)^{k-1}, \ \ldots, \ 2b(2bn+1)^{k-3}, b(2bn+1)^2,}\end{aligned}$$

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$$\left.\overline{2b(2bn+1)^{k-2}, b(2bn+1), 2b(1+2bn)^{k-1}, b, 2(n+b(1+2bn)^k)}\right].$$

The fundamental period in the continued fraction expansion has length $4k+2$.

Since the discovery of Daniel Shanks [11]-[12], there have been a number of examples of families of quadratic surds whose continued fraction expansions have unbounded period length. These include those discovered by Bernstein [2]-[3], Williams [15], Levesque and Rhin [4], Azuhatu [1], van der Poorten [13], and, more recently, Madden [7] and Mollin [8]. Williams paper [15] contains several tables listing surds with long continued fraction expansions, along with the length of their fundamental period.

In this present paper we use a variant of Madden's method to derive several new infinite families of such quadratic surds. Several of these families generalize some of the results of some of the authors cited above.

2. PRELIMINARIES

We first recall some basic properties of continued fractions. For any sequence of numbers $a_0, a_1, \dots,$ define, for $i \geq 0$, the numbers A_i and B_i (the i-th numerator convergent and i-th denominator convergent, respectively, of the continued fraction below) by

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_i} = \frac{A_i}{B_i}.$$

By the correspondence between matrices and continued fractions, we have that

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1} \end{pmatrix}. \tag{2.1}$$

It is also well known that

$$\begin{aligned} A_i &= a_i A_{i-1} + A_{i-2}, \\ B_i &= a_i B_{i-1} + B_{i-2}. \end{aligned} \tag{2.2}$$

See [6], for example, for these basic properties of continued fractions.

Our starting point is the following elementary result.

Lemma 1. *Let $q_0, q_1, \dots, q_1, q_0$ be a finite palindromic sequence of positive integers (with or without a central term) and let*

$$\begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} =: \begin{pmatrix} w & u \\ u & v \end{pmatrix}. \tag{2.3}$$

Then

$$\sqrt{w/v} = [q_0; \overline{q_1, q_2, \dots q_2, q_1, 2q_0}]. \tag{2.4}$$

Proof. As usual, $\overline{q_1, q_2, \dots q_2, q_1, 2q_0}$ means that the sequence of partial quotients $q_1, q_2, \dots q_2, q_1, 2q_0$ is repeated infinitely often. Note that the matrix on the right side of (2.3) is symmetric, since the left side is a symmetric product of symmetric matrices.

Let $\alpha = [q_0; \overline{q_1, q_2, \dots q_2, q_1, 2q_0}]$, so that $\alpha = [q_0; q_1, q_2, \dots q_2, q_1, q_0 + \alpha]$. Then, by (2.3), (2.1) and (2.2), we have that

$$\alpha = \frac{w + \alpha u}{u + \alpha v},$$

so that $\alpha^2 = w/v$ and the result follows. □

We will be primarily interested in the case where w/v is an integer. The next result, although entirely trivial, is also central to what follows.

Lemma 2. *Let D be any complex number and let α and β be any complex numbers such that $\alpha + \beta \neq 0$. Define the matrix P by*

$$P = \begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} -\sqrt{D} & 1 \\ \sqrt{D} & 1 \end{pmatrix}.$$

Then

$$\frac{P_{1,1}}{P_{2,2}} = D.$$

To investigate the fundamental units in the corresponding real quadratic fields $\mathbb{Q}(\sqrt{D})$, there is the following theorem on page 119 of [9]:

Theorem 1. *Let D be a square-free, positive rational integer and let $K = \mathbb{Q}(\sqrt{D})$. Denote by ϵ_0 the fundamental unit of K which exceeds unity, by s the period of the continued fraction expansion for $\sqrt{D}$, and by P/Q the $(s-1)$-th approximant of it.*

If $D \not\equiv 1 \mod 4$ or $D \equiv 1 \mod 8$, then

$$\epsilon_0 = P + Q\sqrt{D}.$$

However, if $D \equiv 5 \mod 8$, then

$$\epsilon_0 = P + Q\sqrt{D}.$$

or

$$\epsilon_0^3 = P + Q\sqrt{D}.$$

Finally, the norm of ϵ_0 is positive if the period s is even and negative otherwise.

This theorem implies the following result.

Proposition 1. *Let D be a non-square positive integer, $D \not\equiv 5(\mod 8)$. Suppose $\sqrt{D} = [q_0; \overline{q_1, \dots, q_1, 2q_0}]$, and that*

$$\begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} w & u \\ u & v \end{pmatrix}$$
$$= \begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} -\sqrt{D} & 1 \\ \sqrt{D} & 1 \end{pmatrix}.$$

Then the fundamental unit in $\mathbb{Q}(\sqrt{D})$ is $2\sqrt{D}\beta$.

Proof. By Theorem 1 and (2.1), the fundamental unit in $\mathbb{Q}(\sqrt{D})$ is $u + v\sqrt{D}$, and the equality of this quantity and $2\sqrt{D}\beta$ follows from comparing corresponding entries in the matrices above. $\square$

Remark: Other approaches can be used to calculate the fundamental unit in the case $D \equiv 5(\mod 8)$, but we do not pursue that here.

We will follow Madden and let $\vec{N}$ denote the sequence $a_1, a_2, \dots, a_j$ whenever

$$N = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},$$

and let $\overleftarrow{N}$ denote the sequence $a_j, a_{j-1}, \dots, a_2, a_1$.

3. Some General Propositions

We next state several variants of a result of Madden from [7]. These general propositions will allow us to construct specific families of long continued fractions in the next section.

Proposition 2. *Let k, u, v, w and r be positive integers such that rw/v is an integer, and let x be a rational such that wx and $2vx$ are integers. Let*

$$C = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose further, for each integer $n \in \{0, 1, \dots k-1\}$, that the matrix N_n defined by

$$N_n := C^{-n} \begin{pmatrix} u & r^{k-1}v \\ w & ru - 2vwx \end{pmatrix} C^n = \begin{pmatrix} u & r^{k-1-n}v \\ r^n w & ru - 2vwx \end{pmatrix}$$

has an expansion of the form

$$N_n = \begin{pmatrix} a_1^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{j_n}^{(n)} & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

where each $a_i^{(n)}$ is a positive integer. Then

$$\begin{aligned} &\sqrt{\frac{wr^k}{v} + w^2x^2} \\ &\quad = [wx; \overline{\vec{N}_{k-1}\vec{N}_{k-2}\dots\vec{N}_1\vec{N}_0, 2vx, \overleftarrow{N}_0, \overleftarrow{N}_1, \dots, \overleftarrow{N}_{k-2}, \overleftarrow{N}_{k-1}, 2wx}]. \end{aligned} \tag{3.2}$$

If $wr^k/v + w^2x^2 \not\equiv 5(\mod 8)$ and is square free, then the fundamental unit in $\mathbb{Q}(wr^k/v + w^2x^2)$ is

$$\frac{w\left(r\,u + v\,\left(-w\,x + \sqrt{\frac{w\,\left(r^k + v\,w\,x^2\right)}{v}}\right)\right)^{2\,k}}{v\,\left(-w\,x + \sqrt{\frac{w\,\left(r^k + v\,w\,x^2\right)}{v}}\right)^2}.$$

Proof. Let $D = wr^k/v + w^2x^2$. We consider the matrix product

$$\begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix} N_{k-1} \ldots N_1 N_0 \begin{pmatrix} 2vx & 1 \\ 1 & 0 \end{pmatrix} N_0^T N_1^T \ldots N_{k-1}^T \begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix}.$$

By the definition of the N_n, this product equals

$$\begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix} C^{-k} (CN_0)^k \begin{pmatrix} 2vx & 1 \\ 1 & 0 \end{pmatrix} ((CN_0)^k)^T C^{-k} \begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix}.$$

Define the matrix M by

$$M := \begin{pmatrix} vx - \frac{v\sqrt{D}}{w} & vx + \frac{v\sqrt{D}}{w} \\ 1 & 1 \end{pmatrix}.$$

One can check that

$$\begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix} C^{-k} = \begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{-wx-\sqrt{D}} & 0 \\ 0 & \frac{1}{-wx+\sqrt{D}} \end{pmatrix} M^{-1};$$

$$(CN_0)^k = M \begin{pmatrix} ru - vwx - v\sqrt{D} & 0 \\ 0 & ru - vwx + v\sqrt{D} \end{pmatrix}^k M^{-1};$$

$$M^{-1} \begin{pmatrix} 2vx & 1 \\ 1 & 0 \end{pmatrix} (M^{-1})^T = \begin{pmatrix} \frac{-w}{2v\sqrt{D}} & 0 \\ 0 & \frac{w}{2v\sqrt{D}} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix} C^{-k} (CN_0)^k \begin{pmatrix} 2vx & 1 \\ 1 & 0 \end{pmatrix} ((CN_0)^k)^T C^{-k} \begin{pmatrix} wx & 1 \\ 1 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{1}{-wx-\sqrt{D}} & 0 \\ 0 & \frac{1}{-wx+\sqrt{D}} \end{pmatrix} \begin{pmatrix} ru - vwx - v\sqrt{D} & 0 \\ 0 & ru - vwx + v\sqrt{D} \end{pmatrix}^k$$
$$\times \begin{pmatrix} \frac{-w}{2v\sqrt{D}} & 0 \\ 0 & \frac{w}{2v\sqrt{D}} \end{pmatrix}$$
$$\times \begin{pmatrix} ru - vwx - v\sqrt{D} & 0 \\ 0 & ru - vwx + v\sqrt{D} \end{pmatrix}^k \begin{pmatrix} \frac{1}{-wx-\sqrt{D}} & 0 \\ 0 & \frac{1}{-wx+\sqrt{D}} \end{pmatrix}$$

$$\times \begin{pmatrix} -\sqrt{D} & 1 \\ \sqrt{D} & 1 \end{pmatrix}.$$

The result follows by Lemmas 1, 2 and Proposition 1. $\square$

The fundamental period of the continued fractions above contain a central partial quotient, namely $2vx$. We next show how to construct long continued fractions which do not have a central partial quotient.

Proposition 3. *Let u, v, x and r be positive integers. Let*

$$C = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose further, for each integer $k \geq 1$ and each integer $n \in \{0, 1, \dots k-1\}$, that the matrix N_n defined by

$$N_n := C^{-n} \begin{pmatrix} u & r^{k-1}v \\ r^k v & ru - 2vx \end{pmatrix} C^n = \begin{pmatrix} u & r^{k-1-n}v \\ r^{k+n}v & ru - 2vx \end{pmatrix}$$

has an expansion of the form

$$N_n = \begin{pmatrix} a_1^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{j_n}^{(n)} & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.3}$$

where each $a_i^{(n)}$ is a positive integer. Then, for each integer $k \geq 1$,

$$\sqrt{r^{2k} + x^2} = [x; \overline{\vec{N}_{k-1} \vec{N}_{k-2} \dots \vec{N}_1 \vec{N}_0, \overleftarrow{N}_0, \overleftarrow{N}_1, \dots, \overleftarrow{N}_{k-2}, \overleftarrow{N}_{k-1}, 2x}]. \tag{3.4}$$

If $r^{2k} + x^2 \not\equiv 5 (\mod 8)$ and is square free, then the fundamental unit in $\mathbb{Q}(\sqrt{r^{2k} + x^2})$ is

$$\left(x + \sqrt{r^{2\,k} + x^2}\right) \left(u + v\, \left(-x + \sqrt{r^{2\,k} + x^2}\right)/r\right)^{2\,k}.$$

Proof. We proceed as in the proof of Proposition 2. Let $D = r^{2k} + x^2$. As above, the definition of N_n implies

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} N_{k-1} \dots N_1 N_0 N_0^T N_1^T \dots N_{k-1}^T \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} C^{-k} (CN_0)^k ((CN_0)^k)^T C^{-k} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}.$$

Define the matrix M by

$$M := \begin{pmatrix} \frac{x-\sqrt{D}}{r^k} & \frac{x+\sqrt{D}}{r^k} \\ 1 & 1 \end{pmatrix}.$$

One can check that

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} C^{-k} = \begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{x-\sqrt{D}}{r^{2k}} & 0 \\ 0 & \frac{x+\sqrt{D}}{r^{2k}} \end{pmatrix} M^{-1};$$

$$(CN_0)^k = M \begin{pmatrix} ru - vx - v\sqrt{D} & 0 \\ 0 & ru - vx + v\sqrt{D} \end{pmatrix}^k M^{-1};$$

$$M^{-1}(M^{-1})^T = \begin{pmatrix} \frac{1}{2} + \frac{x}{2\sqrt{D}} & 0 \\ 0 & \frac{1}{2} - \frac{x}{2\sqrt{D}} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} C^{-k}(CN_0)^k((CN_0)^k)^T C^{-k} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{x-\sqrt{D}}{r^{2k}} & 0 \\ 0 & \frac{x+\sqrt{D}}{r^{2k}} \end{pmatrix} \times \begin{pmatrix} ru - vx - v\sqrt{D} & 0 \\ 0 & ru - vx + v\sqrt{D} \end{pmatrix}^k$$
$$\times \begin{pmatrix} \frac{1}{2} + \frac{x}{2\sqrt{D}} & 0 \\ 0 & \frac{1}{2} - \frac{x}{2\sqrt{D}} \end{pmatrix}$$
$$\times \begin{pmatrix} ru - vx - v\sqrt{D} & 0 \\ 0 & ru - vx + v\sqrt{D} \end{pmatrix}^k \begin{pmatrix} \frac{x-\sqrt{D}}{r^{2k}} & 0 \\ 0 & \frac{x+\sqrt{D}}{r^{2k}} \end{pmatrix}$$
$$\times \begin{pmatrix} -\sqrt{D} & 1 \\ \sqrt{D} & 1 \end{pmatrix},$$

and once again the result follows by Lemmas 1 and 2 and Proposition 1. □

It is also possible to create long continued fractions with no central partial quotient, but with two extra central partial quotients that do not come from $\overrightarrow{N}_0$ and $\overleftarrow{N}_0$.

Proposition 4. *Let u, v, w, q and r be positive integers. Let*

$$C = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose further, for each integer $k \geq 1$ and each integer $n \in \{0, 1, \dots k-1\}$, that the matrix N_n defined by

$$N_n := C^{-n} \begin{pmatrix} u & r^{k-1}v \\ v\left(r^k + 4\,q\,w\right) & r\,u - 2\,q\,r^k\,v - 2\,\left(1 + 4\,q^2\right)\,v\,w \end{pmatrix} C^n$$

$$= \begin{pmatrix} u & r^{k-1-n} v \\ r^n v \left(r^k + 4\,q\,w\right) & r\,u - 2\,q\,r^k\,v - 2\,\left(1 + 4\,q^2\right)\,v\,w \end{pmatrix}$$

has an expansion of the form

$$N_n = \begin{pmatrix} a_1^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2^{(n)} & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_{j_n}^{(n)} & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.5}$$

where each $a_i^{(n)}$ *is a positive integer. Then, for each integer* $k \geq 1$,

$$\sqrt{r^k\,\left(r^k + 4\,q\,w\right) + \left(w + q\,\left(r^k + 4\,q\,w\right)\right)^2} = \tag{3.6}$$

$$\Big[w + q(r^k + 4qw); \overline{\vec{N}_{k-1}\vec{N}_{k-2}\dots\vec{N}_1\vec{N}_0, 2q, 2q, \overleftarrow{N}_0, \overleftarrow{N}_1, \dots, \overleftarrow{N}_{k-2}, \overleftarrow{N}_{k-1}, 2(w + q(r^k + 4qw))} \Big].$$

For ease of notation, set $D = r^k\,\left(r^k + 4\,q\,w\right) + \left(w + q\,\left(r^k + 4\,q\,w\right)\right)^2$ *and* $\gamma = q(r^k + 4qw)$*. Then the fundamental unit in* $\mathbb{Q}(\sqrt{D})$ *is*

$$\frac{\left(u + v\,\left(\sqrt{d} - w - \gamma\right)/r\right)^{2\,k}\,\left(\sqrt{d} - w + \gamma\right)\,\left(\sqrt{d} + w + \gamma\right)^2}{\gamma^2}.$$

Proof. We proceed as in the proof of Propositions 2 and 3. Let

$$D = r^k\,\left(r^k + 4\,q\,w\right) + \left(w + q\,\left(r^k + 4\,q\,w\right)\right)^2.$$

As above, the definition of N_n implies

$$\begin{aligned} &\begin{pmatrix} w + q\,\left(r^k + 4\,q\,w\right) & 1 \\ 1 & 0 \end{pmatrix} N_{k-1} \dots N_1 N_0 \begin{pmatrix} 2q & 1 \\ 1 & 0 \end{pmatrix} \\ &\qquad \times \begin{pmatrix} 2q & 1 \\ 1 & 0 \end{pmatrix} N_0^T N_1^T \dots N_{k-1}^T \begin{pmatrix} w + q\,\left(r^k + 4\,q\,w\right) & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} w + q\,\left(r^k + 4\,q\,w\right) & 1 \\ 1 & 0 \end{pmatrix} C^{-k}(CN_0)^k \begin{pmatrix} 2q & 1 \\ 1 & 0 \end{pmatrix} \\ &\qquad\qquad \times \begin{pmatrix} 2q & 1 \\ 1 & 0 \end{pmatrix} ((CN_0)^k)^T C^{-k} \begin{pmatrix} w + q\,\left(r^k + 4\,q\,w\right) & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Recall that $\gamma = q(r^k + 4qw)$ and define the matrix M by

$$M := \begin{pmatrix} \dfrac{q\left(v(w + \gamma) - \sqrt{D}v\right)}{v\gamma} & \dfrac{q\left((w + \gamma)v + \sqrt{D}v\right)}{v\gamma} \\ & \\ 1 & 1 \end{pmatrix}.$$

One can check (preferably with a computer algebra system like *Mathematica*) that

$$\begin{pmatrix} w+q\,(r^k+4\,q\,w) & 1 \\ 1 & 0 \end{pmatrix} C^{-k}$$
$$= \begin{pmatrix} -\sqrt{D} & \sqrt{D} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{qr^{-k}\left(w+\gamma-\sqrt{D}\right)}{\gamma} & 0 \\ 0 & \dfrac{qr^{-k}\left(w+\gamma+\sqrt{D}\right)}{\gamma} \end{pmatrix} M^{-1};$$

$$(CN_0)^k = M \begin{pmatrix} ru - v\left(w+\gamma+\sqrt{D}\right) & 0 \\ 0 & ru + v\left(-w-\gamma+\sqrt{D}\right) \end{pmatrix}^k M^{-1};$$

$$M^{-1}\begin{pmatrix} 2q & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2q & 1 \\ 1 & 0 \end{pmatrix}(M^{-1})^T = \begin{pmatrix} \dfrac{1}{2} + \dfrac{w-\gamma}{2\sqrt{D}} & 0 \\ 0 & \dfrac{1}{2} - \dfrac{w-\gamma}{2\sqrt{D}} \end{pmatrix}.$$

Thus, as in Propositions 2 and 3, the result follows by Lemmas 1 and 2 and Proposition 1. □

4. LONG CONTINUED FRACTIONS

We next give specific values (in terms of other variables) for some of the variables in the propositions so as to produce explicit long continued fractions. The main problem is to do this in such a way that the matrices N_n satisfy (3.1), (3.3) or (3.5).

From Proposition 2, we have that

$$N_0 = \begin{pmatrix} u & r^{k-1}v \\ w & ru-2vwx \end{pmatrix},$$

so one obvious approach is to initially define u, v, w, x and r from a product of the form

$$N_0 = \begin{pmatrix} u & r^{k-1}v \\ w & ru-2vwx \end{pmatrix} = \begin{pmatrix} a_1^{(0)} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2^{(0)} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{j_0}^{(0)} & 1 \\ 1 & 0 \end{pmatrix}$$

and then specialize the $a_i^{(0)}$ so that each N_n also has an expansion of a similar form. We proceed similarly with Propositions 3 and 4.

We consider one example in detail to illustrate the method. We consider Proposition 2 with

$$N_0 = \begin{pmatrix} u & r^{k-1}v \\ w & ru-2vwx \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix}.$$

Upon comparing $(1,2)$ entries, it can be seen that a must be a multiple of r^{k-1}, so replace a by $r^{k-1}a$ and we then have

$$\begin{pmatrix} u & r^{k-1}v \\ w & ru-2vwx \end{pmatrix} = \begin{pmatrix} 1+r^{k-1}ab & r^{k-1}a \\ b & 1 \end{pmatrix}.$$

Then $v=a$, $w=b$, $u=1+abr^{k-1}$ and $x=(abr^k+r-1)/(2ab)$. The requirements that wx, wr/v and $2vx$ be integers force b and r to have the forms $b=2ma$, $r=1+2ams$, respectively, for some integers m and s. Thus we finally have

$$\begin{aligned} u &= 1+2\,a^2\,m\,(1+2\,a\,m\,s)^{-1+k},\\ v &= a,\\ w &= 2\,a\,m,\\ x &= \frac{s+a\,(1+2\,a\,m\,s)^k}{2\,a}. \end{aligned}$$

With these values, we have that

$$\begin{aligned} \frac{wr^k}{v}+w^2x^2 &= m\left(2\,(1+2\,a\,m\,s)^k + m\left(s+a\,(1+2\,a\,m\,s)^k\right)^2\right),\\ N_n &= \begin{pmatrix} u & r^{k-1-n}v \\ wr^n & ru-2vwx \end{pmatrix}\\ &= \begin{pmatrix} 1+2\,a^2\,m\,(1+2\,a\,m\,s)^{-1+k} & a\,(1+2\,a\,m\,s)^{-1+k-n} \\ 2\,a\,m\,(1+2\,a\,m\,s)^n & 1 \end{pmatrix}\\ &= \begin{pmatrix} a\,(1+2\,a\,m\,s)^{-1+k-n} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2\,a\,m\,(1+2\,a\,m\,s)^n & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\vec{N_n} = a\,(1+2\,a\,m\,s)^{-1+k-n},\, 2\,a\,m\,(1+2\,a\,m\,s)^n.$$

For a non-square integer D, let $l(D)$ denote the length of the fundamental period in the regular continued fraction expansion of $\sqrt{D}$. We have proved the following theorem.

Theorem 2. *Let a, m, s and k be positive integers. Set*

$$D := m\left(2\,(1+2\,a\,m\,s)^k + m\left(s+a\,(1+2\,a\,m\,s)^k\right)^2\right).$$

Then $l(D)=4k+2$ and

$$\sqrt{D} = \Big[m\,(s+a\,(1+2\,a\,m\,s)^k);\overline{a,\ 2\,a\,m\,(1+2\,a\,m\,s)^{k-1},}$$

$$\overline{a(2ams+1),\, 2am(2ams+1)^{k-2},}$$
$$\overline{a(2ams+1)^2,\, 2am(2ams+1)^{k-3},}$$
$$\ldots$$
$$\overline{a(1+2ams)^{k-1},\, 2am,}$$
$$\overline{s+a(2ams+1)^{k},}$$
$$\overline{2am,\, a(1+2ams)^{k-1},}$$
$$\ldots,$$
$$\overline{2am(2ams+1)^{k-3},\, a(2ams+1)^2,}$$
$$\overline{2am(2ams+1)^{k-2},\, a(2ams+1),}$$
$$\overline{2\,a\,m\,(1+2\,a\,m\,s)^{k-1},\, a,}$$
$$\left.\overline{2m\,(s+a\,(1+2\,a\,m\,s)^{k})}\right].$$

This long continued fraction generalizes Madden's first example in Section 3 of [7], where Madden's continued fraction is the case $m = 1$ of the continued fraction above (This continued fraction of Madden is also given in row 1 of Table 3 in Williams paper [15]). The case $m = 1$, $a = 1$ gives Bernstein's Theorem 3 from [2].

Theorem 3. *Let $a > 1$, m, s and k be positive integers. Set*

$$D := m\left(2\,(2a\,m\,s-1)^{k} + m\left(s - a\,(2a\,m\,s-1)^{k}\right)^{2}\right).$$

Then $l(D) = 6k + 2$ and

$$\sqrt{D} = \Big[m\,(-s + a\,(2a\,m\,s-1)^{k});\, \overline{a-1,\, 1,\, 2\,a\,m\,(2a\,m\,s-1)^{k-1}-1,}$$
$$\overline{a(2ams-1)-1,\, 1,\, 2am(2ams-1)^{k-2}-1,}$$
$$\overline{a(2ams-1)^2-1,\, 1,\, 2am(2ams-1)^{k-3}-1,}$$
$$\ldots$$
$$\overline{a(-1+2ams)^{k-1}-1,\, 1,\, 2am-1,}$$
$$\overline{-s+a(2ams-1)^{k},}$$
$$\overline{2am-1,\, 1,\, a(1+2ams)^{k-1}-1,}$$
$$\ldots,$$
$$\overline{2am(2ams-1)^{k-3}-1,\, 1,\, a(2ams-1)^2-1,}$$
$$\overline{2am(2ams-1)^{k-2}-1,\, 1,\, a(1+2ams)-1,}$$

$$\overline{2\,a\,m\,(2a\,m\,s-1)^{k-1}-1,\,1,\,a-1,}$$
$$\left.\overline{2m\,(-s+a\,(2a\,m\,s-1)^{k})}\right].$$

Proof. In Proposition 2, let

$$\begin{aligned}
u &= 2a^2m(2ams-1)^{k-1}-1,\\
v &= a,\\
w &= 2am,\\
r &= 2ams-1,\\
x &= \frac{a(2ams-1)^k-s}{2a}.
\end{aligned}$$

With these values,

$$\begin{aligned}
\frac{wr^k}{v}+w^2x^2 &= m\left(2\,(2a\,m\,s-1)^k+m\left(s-a\,(2a\,m\,s-1)^k\right)^2\right),\\
N_n &= \begin{pmatrix} u & r^{k-1-n}v\\ wr^n & ru-2vwx\end{pmatrix}\\
&= \begin{pmatrix} -1+2\,a^2\,m\,(2a\,m\,s-1)^{-1+k} & a\,(2a\,m\,s-1)^{-1+k-n}\\ 2\,a\,m\,(2a\,m\,s-1)^n & 1\end{pmatrix}\\
&= \begin{pmatrix} a\,(1+2ams)^{k-n-1}-1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}2\,a\,m\,(1+2ams)^n-1 & 1\\ 1 & 0\end{pmatrix}.
\end{aligned}$$

□

This example generalizes Madden's second example (page 130 [7]) (set $m=1$) (Madden's second example can also be found in row 2 of Table 3 in [15]), Theorem 8.1 of Levesque and Rhin [4] (set $a=2$, $m=a$ and $s=1$) and Theorem 10.1 of Levesque and Rhin [4] (set $a=2$, $m=a/2$ and $s=2$).

The example given by van der Poorten in [13], namely

$$D=\left(qa^k-\frac{a+1}{4q}\right)^2+a^k \text{ with } 4q|a+1,$$

follows upon setting $a=2q$, $m=1/2$ and $s=(a+1)/(2q)$.

We can also let a, m and s take negative values in Theorem 2. Any zero- and negative partial quotients in the continued fraction expansion can be removed by the following transformations:

$$\begin{aligned}
&[m,n,0,p,\alpha]=[m,n+p,\alpha],\\
&[m,-n,\alpha]=[m-1,1,n-1,-\alpha].
\end{aligned}\tag{4.1}$$

Out of the eight possible sign combinations for a, m and s, only four lead to distinct polynomials, those in Theorems 2 and 3 (Theorem 3 could also have been proved by replacing s by $-s$ for the case k is even, and replacing m by $-m$ in the case k is odd) and two others.

Our first application of this transformation is to Theorem 3.

Corollary 1. *Let m, s and k be positive integers such that $ms > 1$. Set*

$$D := m\left(2\,(2\,m\,s-1)^k + m\left(s-\ (2\,m\,s-1)^k\right)^2\right).$$

Then $l(D) = 6k-2$ and

$$\begin{aligned}\sqrt{D} = \Big[m\,(-s+\ (2\,m\,s-1)^k)+1;\ &\overline{2\ m\,(2\,m\,s-1)^{k-1}-1,}\\ &\overline{(2ms-1)-1,\ 1,\ 2m(2ms-1)^{k-2}-1,}\\ &\overline{(2ms-1)^2-1,\ 1,\ 2m(2ms-1)^{k-3}-1,}\\ &\qquad\ldots\\ &\overline{(-1+2ms)^{k-1}-1,\ 1,\ 2m-1,}\\ &\overline{-s+(2ms-1)^k,}\\ &\overline{2m-1,\ 1,\ (1+2ms)^{k-1}-1,}\\ &\qquad\ldots,\\ &\overline{2m(2ms-1)^{k-3}-1,\ 1,\ (2ms-1)^2-1,}\\ &\overline{2m(2ms-1)^{k-2}-1,\ 1,\ (1+2ms)-1,}\\ &\overline{2\ m\,(2\,m\,s-1)^{k-1}-1,2m\,(-s+\ (2\,m\,s-1)^k)+2}\Big].\end{aligned}$$

Proof. This follows upon letting $a = 1$ in Theorem 3 and using (4.1) to remove the zeros resulting from the "$a-1$" terms. □

This continued fraction generalizes that in Theorem 2 of Bernstein [2] (set $m = 1$ and $s = a+1$) and also that in Theorem 6.1 of Levesque and Rhin [4] (set $m = 2a$ and $s = 1$).

Theorem 4. *Let $a > 1$, m, s and k be positive integers. Set*

$$D := m\left(m\left(s + a\,(2\,a\,m\,s-1)^k\right)^2 - 2\,(2\,a\,m\,s-1)^k\right).$$

Then $l(D) = 6k+4$ and

(4.2)

$$\begin{aligned}\sqrt{D} = \Big[m\,(s+a\,(2a\,m\,s-1)^k)-1;\ &\overline{1,\ a-1,\ 2\,a\,m\,(2a\,m\,s-1)^{k-1}-1,}\\ &\overline{1,\ a(2ams-1)-1,\ 2am(2ams-1)^{k-2}-1,}\\ &\overline{1,\ a(2ams-1)^2-1,\ 2am(2ams-1)^{k-3}-1,}\end{aligned}$$

$$\begin{aligned}
&\qquad\qquad\dots\\
&\overline{1,\, a(2ams-1)^{k-1}-1,\, 2am-1,}\\
&\overline{1,\, s+a(2ams-1)^{k}-2,\, 1,}\\
&\overline{2am-1,\, a(2ams-1)^{k-1}-1,\, 1,}\\
&\qquad\qquad\dots,\\
&\overline{2am(2ams-1)^{k-3}-1,\, a(2ams-1)^{2}-1,\, 1,}\\
&\overline{2am(2ams-1)^{k-2}-1,\, a(2ams-1)-1,\, 1,}\\
&\overline{2\,a\,m\,(2a\,m\,s-1)^{k-1}-1,\, a-1,\, 1,}\\
&\left.\overline{2m\,(s+a\,(2a\,m\,s-1)^{k})-2}\right].
\end{aligned}$$

Proof. We consider the cases where k is odd and k is even separately. We first consider k odd.

Replace a by $-a$ and k by $2k+1$ in Theorem 2. Then

$$D := m\left(m\left(s+a\,(2\,a\,m\,s-1)^{2k+1}\right)^{2}-2\,(2\,a\,m\,s-1)^{2k+1}\right)$$

and

$$\begin{aligned}
\sqrt{D} = \Big[m\,(s+a\,(2\,a\,m\,s-1)^{2k+1});\, &\overline{-a,\, -2\,a\,m\,(2\,a\,m\,s-1)^{2k},}\\
&\overline{a(2ams-1),\, 2am(2ams-1)^{2k-1},}\\
&\overline{-a(2ams-1)^{2},\, -2am(2ams-1)^{2k-2},}\\
&\qquad\qquad\dots\\
&\overline{a(2ams-1)^{2k-1},\, 2am(2ams-1),}\\
&\overline{-a(2ams-1)^{2k},\, -2am,}\\
&\overline{s+a(2ams-1)^{2k+1},}\\
&\overline{-2am,\, -a(2ams-1)^{2k},}\\
&\overline{2am(2ams-1),\, a(2ams-1)^{2k-1},}\\
&\qquad\qquad\dots,\\
&\overline{-2am(2ams-1)^{2k-2},\, -a(2ams-1)^{2},}\\
&\overline{2am(2ams-1)^{2k-1},\, a(2ams-1),}\\
&\overline{-2\,a\,m\,(2\,a\,m\,s-1)^{2k},\, -a,}\\
&\left.\overline{2m\,(s+a\,(2\,a\,m\,s-1)^{2k+1})}\right].
\end{aligned}$$

We now apply the second identity at (4.1) repeatedly to get that

$$
\begin{aligned}
\sqrt{D} = \Big[& m\,(s + a\,(2a\,m\,s - 1)^{2k+1}) - 1; \overline{1,\, a-1,\, 2\,a\,m\,(2a\,m\,s-1)^{2k} - 1,}\\
& \overline{1,\, a(2ams-1)-1,\, 2am(2ams-1)^{2k-1}-1,}\\
& \overline{1,\, a(2ams-1)^2-1,\, 2am(2ams-1)^{2k-2}-1,}\\
& \qquad \ldots\\
& \overline{1,\, a(2ams-1)^{2k}-1,\, 2am-1,}\\
& \overline{1,\, s+a(2ams-1)^{2k+1}-2,\, 1,}\\
& \overline{1,\, 2am-1,\, a(2ams-1)^{2k}-1,}\\
& \qquad \ldots,\\
& \overline{1,\, 2am(2ams-1)^{2k-2}-1,\, a(2ams-1)^2-1,}\\
& \overline{1,\, 2am(2ams-1)^{2k-1}-1,\, a(2ams-1)-1,}\\
& \overline{1,\, 2\,a\,m\,(2a\,m\,s-1)^{2k}-1,\, a-1,}\\
& \overline{2m\,(s+a\,(2a\,m\,s-1)^{2k+1})-2}\Big].
\end{aligned}
$$

It is clear that $l(D) = 12k + 10$. This proves the theorem for odd k.

We next consider the case where k is even. Replace a by $-a$, s by $-s$, m by $-m$ and k by $2k$ in Theorem 2. Then

$$
D := m\left(m\left(s + a\,(2\,a\,m\,s-1)^{2k}\right)^2 - 2\,(2\,a\,m\,s-1)^{2k}\right)
$$

and

$$
\begin{aligned}
\sqrt{D} = \Big[m\,(s + a\,(2\,a\,m\,s-1)^{2k}); & \overline{-a,\, -2\,a\,m\,(2\,a\,m\,s-1)^{2k-1},}\\
& \overline{a(2ams-1),\, 2am(2ams-1)^{2k-2},}\\
& \overline{-a(2ams-1)^2,\, -2am(2ams-1)^{2k-3},}\\
& \qquad \ldots\\
& \overline{a(2ams-1)^{2k-2},\, 2am(2ams-1),}\\
& \overline{-a(2ams-1)^{2k-1},\, -2am,}\\
& \overline{s+a(2ams-1)^{2k},}\\
& \overline{-2am,\, -a(2ams-1)^{2k-1},}\\
& \overline{2am(2ams-1),\, a(2ams-1)^{2k-2},}\\
& \qquad \ldots,
\end{aligned}
$$

$$\overline{-2am(2ams-1)^{2k-3},\ -a(2ams-1)^2,}$$
$$\overline{2am(2ams-1)^{2k-2},\ a(2ams-1),}$$
$$\overline{-2\,a\,m\,(2\,a\,m\,s-1)^{2k-1},\ -a,}$$
$$\left.\overline{2m\,(s+a\,(2\,a\,m\,s-1)^{2k})}\right].$$

We again apply the second identity at (4.1) repeatedly and the stated continued fraction expansion follows. It is clear that $l(D)=12k+4$ in this case. This completes the proof for even k. □

This theorem generalizes Theorem 7.1 of Levesque and Rhin [4] (set $a=2$, $s=1$ and $m=a$) and Theorem 9.1 of Levesque and Rhin [4] (set $a=2$, $s=2$ and $m=a/2$). Williams example in row 3 of Table 1 in [15] is the case $m=1$ of the continued fraction above.

Remark: It seems likely from the common form of the expansions that there should be an alternative proof of Theorem 4 that covers the even and odd cases simultaneously. However, we do not pursue that here.

Corollary 2. *Let m, s and k be positive integers. Set*

$$D := m\left(m\left(s+\,(2\,m\,s-1)^k\right)^2-2\,(2\,m\,s-1)^k\right).$$

Then $l(D)=6k$ and

$$\sqrt{D}=\Big[m\,(s+(2m\,s-1)^k)-1;\ \overline{2\,m\,(2m\,s-1)^{k-1},} \tag{4.3}$$
$$\overline{1,\ (2ms-1)-1,\ 2m(2ms-1)^{k-2}-1,}$$
$$\overline{1,\ (2ms-1)^2-1,\ 2m(2ms-1)^{k-3}-1,}$$
$$\ldots$$
$$\overline{1,\ (2ms-1)^{k-1}-1,\ 2m-1,}$$
$$\overline{1,\ s+(2ms-1)^k-2,\ 1,}$$
$$\overline{2m-1,\ (2ms-1)^{k-1}-1,\ 1,}$$
$$\ldots,$$
$$\overline{2m(2ms-1)^{k-3}-1,\ (2ms-1)^2-1,\ 1,}$$
$$\overline{2m(2ms-1)^{k-2}-1,\ (2ms-1)-1,\ 1,}$$
$$\left.\overline{2\,m\,(2m\,s-1)^{k-1},\ 2m\,(s+\,(2m\,s-1)^k)-2}\right].$$

Proof. Let $a=1$ in Theorem 4 and use (4.1) to remove the zeros resulting from the "$a-1$" terms. □

This result generalizes that in Theorem 1 of Bernstein [2] (set $s = a + 1$ and $m = 1$) and also that in Theorem 5.1 of Levesque and Rhin [4] (set $s = 1$ and $m = 2a$).

Theorem 5. *Let $a > 2$, m, s and k be positive integers. Set*

$$D := m\left(m\left(-s + a\,(1 + 2\,a\,m\,s)^k\right)^2 - 2\,(1 + 2\,a\,m\,s)^k\right).$$

Then $l(D) = 8k + 4$ and

$$\begin{aligned}\sqrt{D} = \Big[m\,(-s+a\,(1+2\,a\,m\,s)^k) - 1; & \overline{1,\ a-2,\ 1, 2\,a\,m\,(1+2\,a\,m\,s)^{k-1} - 2,}\\ & \overline{1, a(2ams+1) - 2,\ 1, 2am(2ams+1)^{k-2} - 2,}\\ & \overline{1, a(2ams+1)^2 - 2,\ 1, 2am(2ams+1)^{k-3} - 2,}\\ & \qquad \ldots\\ & \overline{1, a(1+2ams)^{k-1} - 2,\ 1, 2am - 2,}\\ & \overline{1, -s + a(2ams+1)^k - 2, 1,}\\ & \overline{2am - 2,\ 1, a(1+2ams)^{k-1} - 2, 1,}\\ & \qquad \ldots,\\ & \overline{2am(2ams+1)^{k-3} - 2,\ 1, a(2ams+1)^2 - 2, 1,}\\ & \overline{2am(2ams+1)^{k-2} - 2,\ 1, a(2ams+1) - 2, 1,}\\ & \overline{2am\,(1+2\,a\,m\,s)^{k-1} - 2,\ 1, a-2, 1,}\\ & \overline{2m\,(-s+a\,(1+2\,a\,m\,s)^k) - 2}\Big].\end{aligned}$$

Proof. Replace a by $-a$ and m by $-m$ in Theorem 2. Then

$$D := m\left(m\left(-s + a\,(2\,a\,m\,s + 1)^k\right)^2 - 2\,(2\,a\,m\,s + 1)^k\right)$$

and

$$\begin{aligned}\sqrt{D} = \Big[m\,(-s + a\,(2\,a\,m\,s+1)^k); & \overline{-a,\ 2\,a\,m\,(2\,a\,m\,s+1)^{k-1},}\\ & \overline{-a(2ams+1),\ 2am(2ams+1)^{k-2},}\\ & \overline{-a(2ams+1)^2,\ 2am(2ams+1)^{k-3},}\\ & \qquad \ldots\\ & \overline{-a(2ams-1)^{k-2},\ 2am(2ams+1),}\\ & \overline{-a(2ams+1)^{k-1},\ 2am,}\\ & \overline{s - a(2ams+1)^k,}\end{aligned}$$

$$\overline{2am,\ -a(2ams+1)^{k-1},}\ \overline{2am(2ams+1),\ -a(2ams+1)^{k-2},}\ \dots,\ \overline{2am(2ams+1)^{k-3},\ -a(2ams+1)^{2},}\ \overline{2am(2ams+1)^{k-2},\ -a(2ams+1),}\ \overline{2\,a\,m\,(2\,a\,m\,s-1)^{k-1},\ -a,}\ \overline{2m\,(-s+a\,(2\,a\,m\,s+1)^{k})}\Big].$$

We again apply the second identity at (4.1) repeatedly and the stated continued fraction expansion follows. It is clear that $l(D) = 8k + 4$ in this case. This completes the proof. $\square$

Williams continued fraction in row 4 of Table 1 in [15] is the case $m = 1$ of the theorem above.

Corollary 3. *Let m, s and $k > 1$ be positive integers such that $ms > 1$. Set*

$$D := m\left(m\left(-s+\ (1+2\,m\,s)^{k}\right)^{2} - 2\,(1+2\,m\,s)^{k}\right).$$

Then $l(D) = 8k$ and

$$\sqrt{D} = \Big[m\,(-s+(1+2\,m\,s)^{k}) - 2;\overline{1,\ 2\,m\,(1+2\,m\,s)^{k-1} - 3,}\ \overline{1,(2ms+1)-2,\ 1,2m(2ms+1)^{k-2}-2,}\ \overline{1,(2ms+1)^{2}-2,\ 1,2m(2ms+1)^{k-3}-2,}\ \cdots\ \overline{1,(1+2ms)^{k-1}-2,\ 1,2m-2,}\ \overline{1,-s+(2ms+1)^{k}-2,1,}\ \overline{2m-2,\ 1,(1+2ms)^{k-1}-2,1,}\ \dots,\ \overline{2m(2ms+1)^{k-3}-2,\ 1,(2ms+1)^{2}-2,1,}\ \overline{2m(2ms+1)^{k-2}-2,\ 1,(2ms+1)-2,1,}\ \overline{2m\,(1+2\,m\,s)^{k-1}-3,\ 1,}\ \overline{2m\,(-s+(1+2\,m\,s)^{k})-4}\Big].$$

Proof. This follows after letting $a = 1$ in Theorem 5 and using (4.1) to remove the resulting zeroes and negatives. $\square$

Corollary 4. *Let $k > 2$ and s be positive integers. Set*

$$D := \Big((1+2\,s)^k - s\Big)^2 - 2\,(1+2\,s)^k.$$

Then $l(D) = 8k-4$ and

$$\begin{aligned}\sqrt{D} = \Big[&(1+2\,s)^k - s - 2; \overline{1,\, 2\,(1+2\,s)^{k-1} - 3,}\\ &\overline{1, (2s+1) - 2,\; 1, 2(2s+1)^{k-2} - 2,}\\ &\overline{1, (2s+1)^2 - 2,\; 1, 2(2s+1)^{k-3} - 2,}\\ &\qquad\ldots\\ &\overline{1, (1+2s)^{k-2} - 2,\; 1,\; 2(2s+1) - 2}\\ &\overline{1, (1+2s)^{k-1} - 2,\; 2,}\\ &\overline{-s + (2s+1)^k - 2,}\\ &\overline{2, (1+2s)^{k-1} - 2, 1,}\\ &\overline{2(2s+1) - 2,\; 1,\; , (1+2s)^{k-1} - 2, 1,}\\ &\qquad\ldots,\\ &\overline{2(2s+1)^{k-3} - 2,\; 1, (2s+1)^2 - 2, 1,}\\ &\overline{2(2s+1)^{k-2} - 2,\; 1, (2s+1) - 2, 1,}\\ &\overline{2(1+2\,s)^{k-1} - 3,\; 1,}\\ &\overline{2(s + (1+2\,s)^k) - 4}\Big].\end{aligned}$$

Proof. This follows after letting $m = 1$ in Corollary 3 and using (4.1) to remove the zero resulting from the “$2m-2$” terms. ☐

The result above is Theorem 4 in Bernstein [2].

Corollary 5. *Let m, s and $k > 1$ be positive integers. Set*

$$D := m\left(m\left(-s + 2\,(1+4m\,s)^k\right)^2 - 2\,(1+4m\,s)^k\right).$$

Then $l(D) = 8k$ and

$$\begin{aligned}\sqrt{D} = \Big[&m\,(-s+2\,(1+4m\,s)^k) - 1; \overline{2, 4m\,(1+4m\,s)^{k-1} - 2,}\\ &\overline{1, 2(4ms+1) - 2,\; 1, 4m(4ms+1)^{k-2} - 2,}\\ &\overline{1, 2(4ms+1)^2 - 2,\; 1, 4m(4ms+1)^{k-3} - 2,}\\ &\qquad\ldots\end{aligned}$$

$$\overline{1, 2(1+4ms)^{k-1}-2,\ 1, 4m-2,}$$
$$\overline{1, -s+2(4ms+1)^{k}-2, 1,}$$
$$\overline{4m-2,\ 1, 2(1+4ms)^{k-1}-2, 1,}$$
$$\ldots,$$
$$\overline{4m(4ms+1)^{k-3}-2,\ 1, 2(4ms+1)^{2}-2, 1,}$$
$$\overline{4m(4ms+1)^{k-2}-2,\ 1, 2(4ms+1)-2, 1,}$$
$$\overline{4m\,(1+4m\,s)^{k-1}-2,\ 2,}$$
$$\left.\overline{2m\,(-s+2\,(1+4m\,s)^{k})-2}\right].$$

Proof. This follows after letting $a = 2$ in Theorem 5 and using (4.1) to remove the zero resulting from the "$a - 2$" terms. $\square$

We next use Propositions 3 and 4 to construct families of long continued fractions with no central partial quotient in the fundamental period.

Theorem 6. *Let b, s and k be positive integers. Set*

$$D := (4bs+1)^{k} + \Big(b(4bs+1)^{k}+s\Big)^{2}.$$

Then $l(D) = 2k+1$ and

$$\sqrt{D} = \Big[b(4bs+1)^{k}+s;\ \overline{2b,\quad 2b(4bs+1)^{k-1},}$$
$$\overline{2b(4bs+1),\ 2b(4bs+1)^{k-2},}$$
$$\overline{2b(4bs+1)^{2},\ 2b(4bs+1)^{k-3},}$$
$$\ldots$$
$$\overline{2b(4bs+1)^{\lfloor(k-1)/2\rfloor},\ 2b(4bs+1)^{k-1-\lfloor(k-1)/2\rfloor},}$$
$$\ldots,$$
$$\overline{2b(4bs+1)^{k-3},\ 2b(4bs+1)^{2},}$$
$$\overline{2b(4bs+1)^{k-2},\ 2b(4bs+1),}$$
$$\left.\overline{2b(4bs+1)^{k-1},\ 2b,\ 2(b(4bs+1)^{k}+s)}\right].$$

Here we understand the mid-point of the period to be just after the $2b(4bs+1)^{\lfloor(k-1)/2\rfloor}$ term, if k is odd, and just after the $2b(4bs+1)^{k-1-\lfloor(k-1)/2\rfloor}$ term, if k is even.

Proof. We first prove this for even k. In Proposition 3 set $v = 2b$, $r = 1+4bs$, $u = 1+4b^{2}(1+4bs)^{2k-1}$ and $x = s+b(1+4bs)^{2k}$. Then

$$D = (4bs+1)^{2k} + \Big(b(4bs+1)^{2k}+s\Big)^{2}.$$

With these values also,

$$N_n = \begin{pmatrix} u & r^{k-1-n}v \\ r^{k+n}v & ru-2vx \end{pmatrix} = \begin{pmatrix} 2b(1+4bs)^{k-1-n} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2b(1+4bs)^{k+n} & 1 \\ 1 & 0 \end{pmatrix},$$

and so $\vec{N}_n = 2b(1+4bs)^{k-1-n},\ 2b(1+4bs)^{k+n}$. Thus

$$\begin{aligned}\sqrt{D} = \Big[b(4bs+1)^{2k}+s;\ & \overline{2b,\quad 2b(4bs+1)^{2k-1},} \\ & \overline{2b(4bs+1),\ 2b(4bs+1)^{2k-2},} \\ & \overline{2b(4bs+1)^2,\ 2b(4bs+1)^{2k-3},} \\ & \qquad\qquad \dots \\ & \overline{2b(4bs+1)^{k-1},\ 2b(4bs+1)^{k},} \\ & \overline{2b(4bs+1)^{k},\ 2b(4bs+1)^{k-1},} \\ & \qquad\qquad \dots, \\ & \overline{2b(4bs+1)^{2k-3},\ 2b(4bs+1)^{2},} \\ & \overline{2b(4bs+1)^{2k-2},\ 2b(4bs+1),} \\ & \overline{2b(4bs+1)^{2k-1},\ 2b,\ 2(b(4bs+1)^{2k}+s)}\Big],\end{aligned}$$

and the result follows for even k.

For odd k we use Proposition 4 above, where we set $v = 2b$, $r = 1+4bs$, $u = 1+4b^2(1+4bs)^{2k}$, $q = b(1+4bs)^k$ and $w = s$. Then

$$D = (4bs+1)^{2k+1} + \Big(b(4bs+1)^{2k+1}+s\Big)^2.$$

With these values also,

$$\begin{aligned} N_n &= \begin{pmatrix} u & r^{k-1-n}v \\ r^n v\,\left(r^k+4\,q\,w\right) & r\,u - 2\,q\,r^k\,v - 2\,\left(1+4\,q^2\right)\,v\,w \end{pmatrix} \\ &= \begin{pmatrix} 2b(1+4bs)^{k-1-n} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2b(1+4bs)^{k+n+1} & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and so $\vec{N}_n = 2b(1+4bs)^{k-1-n},\ 2b(1+4bs)^{k+n+1}$. Thus

$$\begin{aligned}\sqrt{D} = \Big[b(4bs+1)^{2k+1}+s;\ & \overline{2b,\quad 2b(4bs+1)^{2k},} \\ & \overline{2b(4bs+1),\ 2b(4bs+1)^{2k-1},} \\ & \overline{2b(4bs+1)^2,\ 2b(4bs+1)^{2k-2},} \\ & \qquad\qquad \dots \\ & \overline{2b(4bs+1)^{k-1},\ 2b(4bs+1)^{k+1},} \\ & \overline{2b(4bs+1)^{k},\ 2b(4bs+1)^{k},} \\ & \overline{2b(4bs+1)^{k+1},\ 2b(4bs+1)^{k-1},}\end{aligned}$$

$$\ldots,$$
$$\overline{2b(4bs+1)^{2k-2},\, 2b(4bs+1)^2,}$$
$$\overline{2b(4bs+1)^{2k-1},\, 2b(4bs+1),}$$
$$\left.\overline{2b(4bs+1)^{2k},\ 2b,\ 2(b(4bs+1)^{2k+1}+s)}\right],$$

and the result again follows. □

The result above (without the explicit continued fraction expansion) appears in row 1 of Table 2 in [15].

As previously, we can let b or r take negative integral values and produce a new long continued fraction.

Corollary 6. *Let b, s and k be positive integers. Set*

$$D := (4bs-1)^{2k} + \Big(b(4bs-1)^{2k} - s\Big)^2.$$

Then $l(D) = 6k+1$ and

$$\sqrt{D} = \Big[b(4bs-1)^{2k} - s;\ \overline{2b-1,\ 1, 2b(4bs-1)^{2k-1} - 1,}$$
$$\overline{2b(4bs-1)-1,\ 1, 2b(4bs-1)^{2k-2}-1,}$$
$$\overline{2b(4bs-1)^2-1,\ 1, 2b(4bs-1)^{2k-3}-1,}$$
$$\ldots$$
$$\overline{2b(4bs-1)^{k-1}-1,\ 1, 2b(4bs-1)^{k}-1,}$$
$$\overline{2b(4bs-1)^{k}-1,\ 1, 2b(4bs-1)^{k-1}-1,}$$
$$\ldots,$$
$$\overline{2b(4bs-1)^{2k-3}-1,\ 2b(4bs-1)^2-1,}$$
$$\overline{2b(4bs-1)^{2k-2}-1,\ 1, 2b(4bs-1)-1,}$$
$$\left.\overline{2b(4bs-1)^{2k-1}-1,\ 1, 2b-1,\ 2(b(4bs-1)^{2k}-s)}\right]$$

Proof. This follows from Theorem 6 in the case k is even, after replacing s by $-s$ and using (4.1) to remove the resulting negative partial quotients from the resulting continued fraction expansion □

Corollary 7. *Let b, s and k be positive integers. Set*

$$D := \Big(b(4bs-1)^{2k+1} + s\Big)^2 - (4bs-1)^{2k+1}.$$

Then $l(D) = 6k+5$ and

$$\sqrt{D} = \Big[b(4bs-1)^{2k+1} + s - 1;\ \overline{1,\ 2b-1,\ 2b(4bs-1)^{2k}-1,}$$
$$\overline{1,\ 2b(4bs-1)-1,\ 2b(4bs-1)^{2k-1}-1,}$$

$$
\begin{aligned}
&\overline{1,\, 2b(4bs-1)^2-1,\, 2b(4bs-1)^{2k-2}-1,}\\
&\qquad\qquad \dots\\
&\overline{1,\, 2b(4bs-1)^{k-1}-1,\, 2b(4bs-1)^{k+1}-1,}\\
&\overline{1,\, 2b(4bs-1)^{k}-1,\, 2b(4bs-1)^{k}-1, 1,}\\
&\overline{2b(4bs-1)^{k+1}-1,\, 2b(4bs-1)^{k-1}-1, 1,}\\
&\qquad\qquad \dots,\\
&\overline{2b(4bs-1)^{2k-2}-1,\, 2b(4bs-1)^{2}-1, 1,}\\
&\overline{2b(4bs-1)^{2k-1}-1,\, 2b(4bs-1)-1, 1,}\\
&\left.\overline{2b(4bs-1)^{2k}-1,\ 2b-1,\, 1, 2(b(4bs-1)^{2k+1}+s-2)}\right].
\end{aligned}
$$

Proof. This follows from Theorem 6 in the case k is odd, after replacing b by $-b$ and using (4.1) to remove the resulting negative partial quotients from the resulting continued fraction expansion. □

Remark: Statements similar to those in Corollaries 6 and 7 hold if $2k$ is replaced by $2k+1$, but these results cannot be derived from Theorem 6.

5. Fundamental Units in Real Quadratic Fields

It is straightforward in many cases to compute the fundamental units in the real quadratic fields corresponding to the surds in the various theorems and corollaries. In what follows, we assume D is square free and $D \not\equiv 5($ mod $8)$.

In Theorem 2, for example, where

$$D = m(2(1+2ams)^k + m(s + a(1+2ams)^k)^2),$$

it follows directly from Proposition 2 that the fundamental unit in $\mathbb{Q}(\sqrt{D})$ is

$$\frac{2\,m\left(1+a\left(\sqrt{D}+m\left(s+a\left(1+2\,a\,m\,s\right)^k\right)\right)\right)^{2\,k}}{\left(\sqrt{D}-m\left(s+a\left(1+2\,a\,m\,s\right)^k\right)\right)^2}.$$

With $m=3$, $a=5$, $s=7$ and $k=2$, for example, we get $D = 446005190022$ and that the fundamental unit in $\mathbb{Q}(\sqrt{446005190022})$ is (after simplifying)

$$
\begin{aligned}
&149199899813252915906267542273\\
&\qquad\qquad\qquad + 223407925198820626278032\sqrt{446005190022}.
\end{aligned}
$$

Likewise, in Theorem 6 (for even k), where

$$D = (1+4bs)^{2k} + (s + b(1+4bs)^{2k})^2,$$

Proposition 3 gives that the fundamental unit in $\mathbb{Q}(\sqrt{D})$ is

$$\frac{1}{(1+4bs)^{2k}}\left(s+b(1+4bs)^{2k}+\sqrt{(1+4bs)^{2k}+\left(s+b(1+4bs)^{2k}\right)^2}\right)$$
$$\times\Bigg(-2\,b\,\left(s+b\,(1+4\,b\,s)^{2\,k}\right)+(1+4\,b\,s)\,\left(1+4\,b^2\,(1+4\,b\,s)^{-1+2\,k}\right)$$
$$+2\,b\,\sqrt{(1+4\,b\,s)^{2\,k}+\left(s+b\,(1+4\,b\,s)^{2\,k}\right)^2}\,\Bigg)^{2\,k}.$$

Setting $s=3$, $b=5$, and $k=2$, for example, gives $D=4792683254153105$ and that the fundamental unit in $\mathbb{Q}(\sqrt{4792683254153105})$ is

$$183758510292882607664916360256981148448+$$
$$265434944781468068474213871001\sqrt{4792683254153105}.$$

For our last example, we also consider Theorem 6 (for odd k), where

$$D=(1+4bs)^{2k+1}+(s+b(1+4bs)^{2k+1})^2,$$

Proposition 4 gives that the fundamental unit in $\mathbb{Q}(\sqrt{D})$ is

$$\frac{1}{(1+4\,b\,s)^{2\,k}}\left(1+2\,b\,\sqrt{D}+2\,b\,s+2\,b^2\,(1+4\,b\,s)^{1+2\,k}\right)^{2\,k}\times$$
$$\left(\sqrt{D}+s+4\,b^3\,(1+4\,b\,s)^{1+4\,k}+b\,(1+4\,b\,s)^{2\,k}\,\left(3+4\,b\,\left(\sqrt{D}+2\,s\right)\right)\right).$$

Letting $s=5$, $b=2$, and $k=1$ gives $D=19001864330$ and shows that the fundamental unit in $\mathbb{Q}(\sqrt{19001864330})$ is

$$2682318982172034563+19458632525153\sqrt{19001864330}.$$

6. Concluding Remarks

While considering the various long continued fractions in the papers by Bernstein [2]-[3] and Levesque and Rhin [4], and trying to see if any more of the continued fractions in these papers could be generalized by the methods in the present paper, we were led to a new construction.

This new construction succeeds where the methods in the present paper fail, in that it allowed us to generalize some more of the continued fractions in the papers mentioned above.

We will investigate this new construction in a subsequent paper.

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