CONTINUED FRACTIONS AND GENERALIZATIONS WITH MANY LIMITS: A SURVEY.

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Abstract. There are infinite processes (matrix products, continued fractions, (r, s)-matrix continued fractions, recurrence sequences) which, under certain circumstances, do not converge but instead diverge in a very predictable way.

We give a survey of results in this area, focusing on recent results of the authors.

1. Introduction

Consider the following recurrence:

$$x_{n+1} = \frac{4}{3} - \frac{1}{x_n}.$$

Taking $1/\infty$ to be 0 and vice versa, then regardless of the initial (real) value of this sequence, it is an interesting fact that the sequence is dense in $\mathbb{R}$. The proof is illuminating.

Take $x_0 = 4/3$ and view x_n as n'th approximant of the continued fraction:

$$4/3 - \frac{1}{4/3} - \frac{1}{4/3} - \frac{1}{4/3} - \frac{1}{4/3} - \cdots. \tag{1}$$

Then, from the standard theorem on the recurrence for convergents of a continued fraction, the n'th numerator and denominator convergents of this continued fraction, A_n and B_n respectively, must both satisfy the linear recurrence relation

$$Y_n = \frac{4}{3}Y_{n-1} - Y_{n-2},$$

but with different initial conditions.

Now, the characteristic roots of this equation are $\alpha = 2/3 + i\sqrt{5}/3$, and $\beta = 2/3 - i\sqrt{5}/3$. Thus from the usual formula for solving linear recurrences, the exact formula for x_n is

$$x_n = \frac{A_n}{B_n} = \frac{a\alpha^n + b\beta^n}{c\alpha^n + d\beta^n} = \frac{a\lambda^n + b}{c\lambda^n + d},$$

where a, b, c, and d are some complex constants and $\lambda = \alpha/\beta$. Notice that λ is a number on the unit circle and is not a root of unity, so that λ^n is dense on the unit circle. The conclusion follows by noting that the linear fractional transformation

$$z \mapsto \frac{az + b}{cz + d}$$

must take the unit circle to $\mathbb{R}$, since the values of the sequence x_n are real.

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After seeing this argument, one is tempted to write down the amusing identity

$$\mathbb{R} = 4/3 - \frac{1}{4/3} - \frac{1}{4/3} - \frac{1}{4/3} - \frac{1}{4/3} - \cdots.$$

This identity is true so long as one interprets the value of the continued fraction to be the set of limits of subsequences of its sequence of approximants.

Another motivating example of our work is the following theorem, one of the oldest in the analytic theory of continued fractions [6]:

Theorem 1.1. *(Stern-Stolz) Let the sequence* $\{b_n\}$ *satisfy* $\sum |b_n| < \infty$. *Then*

$$b_0 + K_{n=1}^{\infty} \frac{1}{b_n}$$

diverges. In fact, for $p = 0, 1$,

$$\lim_{n\to\infty} P_{2n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{2n+p} = B_p \neq \infty,$$

and

$$A_1 B_0 - A_0 B_1 = 1.$$

The Stern-Stolz theorem gives a general class of continued fractions each of which tend to two different limits, respectively A_0/B_0, and A_1/B_1. Here and throughout we assume the limits for continued fractions are in $\widehat{\mathbb{C}}$. This makes sense because continued fractions can be viewed as the composition of linear fractional transformations and such functions have $\widehat{\mathbb{C}}$ as their natural domain and codomain.

Before leaving the Stern-Stolz theorem, we wish to remark that although the theorem is usually termed a "divergence theorem", this terminology is a bit misleading; the theorem actually shows that although the continued fractions of this form diverge, they do so by tending to two limits in a precisely controlled way. In this paper we study extensions of this phenomenon and investigate just how far one can go in this direction. Thus, although throughout this paper we refer to certain of our results as "divergence" theorems, most of them actually give explicit results about convergent subsequences.

A special case of the Stern-Stolz theorem gives a result on the famous Rogers-Ramanujan continued fraction:

$$1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} \cdots, \tag{2}$$

The Stern-Stolz theorem gives that for $|q| > 1$ the even and odd approximants of this continued fraction tend to two limiting functions. To see this, observe that by the standard equivalence transformation for continued fractions, (2) is equal to

$$1 + \frac{1}{1/q} + \frac{1}{1/q} + \frac{1}{1/q^2} + \frac{1}{1/q^2} \cdots + \frac{1}{1/q^n} + \frac{1}{1/q^n} \cdots.$$

The Stern-Stolz theorem, however does not apply to the following continued fraction given by Ramanujan:

$$\frac{-1}{1+q} + \frac{-1}{1+q^2} + \frac{-1}{1+q^3} + \cdots. \tag{3}$$

Recently in [1] Andrews, Berndt, *et al.* proved a claim made by Ramanujan in his lost notebook ([9], p.45) about (3). To describe Ramanujan's claim, we first need some notation.

Throughout take $q \in \mathbb{C}$ with $|q| < 1$. The following standard notation for q-products will also be employed:

$$(a)_0 := (a;q)_0 := 1, \qquad (a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1 - a\,q^k), \qquad \text{if } n \geq 1,$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - a\,q^k), \qquad |q| < 1.$$

Set $\omega = e^{2\pi i/3}$. Ramanujan's claim was that, for $|q| < 1$,

$$\lim_{n\to\infty}\left(\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n+a}\right) = -\omega^2\left(\frac{\Omega - \omega^{n+1}}{\Omega - \omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}, \tag{4}$$

where

$$\Omega := \frac{1 - a\omega^2}{1 - a\omega}\frac{(\omega^2 q, q)_\infty}{(\omega q, q)_\infty}.$$

Ramanujan's notation is confusing, but what his claim means is that the limit exists as $n \to \infty$ in each of the three congruence classes modulo 3, and that the limit is given by the expression on the right side of (4). Also, the appearance of the variable a in this formula is a bit of a red herring; from elementary properties of continued fractions, one can derive the result for general a from the $a = 0$ case.

Now (1) is different from the other examples in that it has subsequences of approximants tending to infinitely many limits. Nevertheless, all of the examples above, including (1), are special cases of a general result on continued fractions (Theorem 4.1 below). To deal with both of these situations we introduce the notion of the *limit set* of a sequence.

The *limit set* of the sequence is defined to be the set of all limits of convergent subsequences. Limit sets should not be confused with sets of limit points. Thus, for example, the sequence $\{1, 1, 1, \dots\}$ has limit set $\{1\}$ although the set of limit (accumulation) points of the set of values of the sequence is empty. Limit sets need to be introduced so that sequences with constant subsequences will have the values of these subsequences included among the possible limits. Certain periodic continued fractions have this property. To avoid confusion we designate the limit set of a sequence $\{s_n\}_{n\geq 1}$ by $l.s.(s_n)$.

Our initial research [2] dealt with cases in which the limit set was finite. In [3] we extended our methods to give a uniform treatment of finite and infinite cases. In fact, in [3], we studied asymptotics for approximants for infinite matrix products, continued fractions, and recurrence relations of Poincaré type. Limit set information easily follows from the asymptotics.

In the papers [2] and [3], the authors studied limit sets in the specific context of sequences of the form

$$f\left(\prod_{i=1}^{n} D_i\right),$$

where D_i is a sequence of complex matrices and f is a function with values in some compact metric space.

2. Definitions, Notation and Terminology

Limit set equalities in this paper arise from the situation

$$\lim_{n\to\infty} d(s_n, t_n) = 0$$

in some metric space (X, d). Accordingly, it makes sense to define the equivalence relation $\sim$ on sequences in X by $\{s_n\} \sim \{t_n\} \iff \lim_{n\to\infty} d(s_n, t_n) = 0$. In this situation we refer to sequences $\{s_n\}$ and $\{t_n\}$ as being asymptotic to each other. Abusing notation, we often write $s_n \sim t_n$ in place of $\{s_n\} \sim \{t_n\}$. More generally, we frequently write sequences without braces when it is clear from context that we are speaking of a sequence, and not the nth term.

Let $M_d(\mathbb{C})$ denote the set of $d \times d$ matrices of complex numbers topologised using the l_∞ norm, denoted by $||\cdot||$. Let I denote the identity matrix. When we use product notation for matrices, the product is taken from left to right; thus

$$\prod_{i=1}^{n} A_i := A_1 A_2 \cdots A_n.$$

An infinite continued fraction

$$K_{n=1}^{\infty} \frac{a_n}{b_n} := \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{5}$$

is said to converge if

$$\lim_{n\to\infty} \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}$$

exists in $\widehat{\mathbb{C}}$. Let $\{\omega_n\}$ be a sequence of complex numbers. If

$$\lim_{n\to\infty} \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n + \omega_n}$$

exist, then this limit is called the *modified limit of $K_{n=1}^{\infty} a_n/b_n$ with respect to the sequence* $\{\omega_n\}$. Detailed discussions of modified continued fractions as well as further pointers to the literature are given in [6].

We follow the common convention in analysis of denoting the group of points on the unit circle by $\mathbb{T}$, or by $\mathbb{T}_\infty$, and its subgroup of roots of unity of order m, m finite, by $\mathbb{T}_m$. (Note: $\mathbb{T}_\infty$ often denotes the group of all roots of unity; here it denotes the whole circle group.)

3. Theorems of Infinite Matrix Products

The classic theorem on the convergence of infinite products of matrices seems first to have been given clearly by Wedderburn [10].

Proposition 1. *(Wedderburn* [10, 11]*) Let $A_i \in M_d(\mathbb{C})$ for $i \geq 1$. Then $\sum_{i\geq 1} ||A_i|| < \infty$ implies that $\prod_{i\geq 1}(I + A_i)$ converges in $M_d(\mathbb{C})$.*

In [2], our initial motivation was to generalize the Ramanujan continued fraction with three limits (4) to a continued fraction with m limits, $m \geq 3$. This led us to consider infinite sequences of matrices converging to 2×2 matrices with eigenvalues which were distinct roots of unity, and to examine the divergence of the corresponding infinite matrix product.

This in turn led us to consider the more general case of infinite sequences of $p\times p$ matrices, $p \geq 2$, with similar properties. In [2] we proved the following result.

Proposition 2. *Let $p \geq 2$ be an integer and let M be a $p \times p$ matrix that is diagonalizable and whose eigenvalues are roots of unity. Let I denote the $p \times p$ identity matrix and let m be the least positive integer such that*

$$M^m = I.$$

For a $p \times p$ matrix G, let

$$||G||_\infty = \max_{1 \leq i,j \leq p} |G^{(i,j)}|,$$

where $G^{(i,j)}$ denotes the element of G in row i and column j. Suppose $\{D_n\}_{n=1}^\infty$ is a sequence of matrices such that

$$\sum_{n=1}^\infty ||D_n - M||_\infty < \infty.$$

Then

$$F := \lim_{k \to \infty} \prod_{n=1}^{km} D_n$$

exists. Here the matrix product means either $D_1 D_2 \dots$ or $\dots D_2 D_1$. Further, for each j, $0 \leq j \leq m-1$,

$$\lim_{k \to \infty} \prod_{n=1}^{km+j} D_n = M^j F \text{ or } F M^j,$$

depending on whether the products are taken to the left or right.

A natural progression was to replace the matrix M in the proposition above with a sequence of matrices $\{M_i\}$. In [3] we proved the following result.

Theorem 1. *Suppose $\{M_i\}$ and $\{D_i\}$ are sequences of complex matrices such that the two sequences (for $\epsilon = \pm 1$)*

$$\left\| \left(\prod_{i=1}^n M_i \right)^\epsilon \right\| \tag{6}$$

are bounded and

$$\sum_{i \geq 1} \|D_i - M_i\| < \infty. \tag{7}$$

Then

$$F := \lim_{n \to \infty} \left(\prod_{i=1}^n D_i \right) \left(\prod_{i=1}^n M_i \right)^{-1} \tag{8}$$

exists and $\det(F) \neq 0$ if and only if $\det(D_i) \neq 0$ for all $i \geq 1$.

As sequences

$$\prod_{i=1}^n D_i \sim F \prod_{i=1}^n M_i. \tag{9}$$

More generally, let f be a continuous function from the domain

$$\overline{\left\{ F \prod_{i=1}^n M_i : n \geq h \right\}} \cup \bigcup_{n \geq h} \left\{ \prod_{i=1}^n D_i \right\},$$

for some integer $h \geq 1$, into a metric space G. Then the domain of f is compact in $M_d(\mathbb{C})$ and $f(\prod_{i=1}^n D_i) \sim f(F\prod_{i=1}^n M_i)$. Finally

$$l.s.\left(\prod_{i=1}^{n} D_i\right) = l.s.\left(F\prod_{i=1}^{n} M_i\right), \tag{10}$$

and

$$l.s.\left(f\left(\prod_{i=1}^{n} D_i\right)\right) = l.s.\left(f\left(F\prod_{i=1}^{n} M_i\right)\right). \tag{11}$$

Theorem 1 had several interesting applications to certain classes of continued fractions, recurrence sequences, and (r,s)-matrix continued fractions.

4. Theorems on Continued Fractions

We begin by stating our general theorem on the asymptotics and limit sets of the sequence of approximants of a class of continued fractions. The theorem shows that the limit set is a circle (or a finite subset of a circle) on the Riemann sphere. When the limit set is a circle, although the set of approximants approaches all of its points, the approximants usually do not do so in a uniform way.

The following theorem concerns the continued fraction

$$\frac{-\alpha\beta+q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta+q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta+q_n}{\alpha+\beta+p_n}, \tag{12}$$

where the sequences p_n and q_n are absolutely summable and the constants $\alpha \neq \beta$ are points on the unit circle.

Theorem 4.1. *Let $\{p_n\}_{n\geq 1}$, $\{q_n\}_{n\geq 1}$ be complex sequences satisfying*

$$\sum_{n=1}^{\infty} |p_n| < \infty, \qquad \sum_{n=1}^{\infty} |q_n| < \infty.$$

Let α and β satisfy $|\alpha| = |\beta| = 1$, $\alpha \neq \beta$ with the order of $\lambda = \alpha/\beta$ in $\mathbb{T}$ being m (where m may be infinite). Assume that $q_n \neq \alpha\beta$ for any $n \geq 1$. Put

$$f_n(w) := \frac{-\alpha\beta+q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta+q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta+q_n}{\alpha+\beta+p_n+w},$$

so that $f_n := f_n(0)$ is the sequence of approximants of the continued fraction (12)*. Then $f_n \sim h(\lambda^{n+1})$ so that $l.s.(f_n) = h(\mathbb{T}_m)$, where*

$$h(z) = \frac{az+b}{cz+d},$$

with the constants $a, b, c, d \in \mathbb{C}$ given by the (existent) limits

$$\begin{aligned} a &= \lim_{n\to\infty} \alpha^{-n}(P_n - \beta P_{n-1}), \\ b &= -\lim_{n\to\infty} \beta^{-n}(P_n - \alpha P_{n-1}), \\ c &= \lim_{n\to\infty} \alpha^{-n}(Q_n - \beta Q_{n-1}), \\ d &= -\lim_{n\to\infty} \beta^{-n}(Q_n - \alpha Q_{n-1}), \end{aligned} \tag{13}$$

where P_n and Q_n are the nth convergents of the continued fraction (12)*. Moreover,*

$$\det(h) = ad - bc = (\beta - \alpha)\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\alpha\beta}\right) \neq 0, \tag{14}$$

and the following identities involving modified versions of (12) *hold in* $\widehat{\mathbb{C}}$*:*

$$\begin{aligned} h(\infty) &= \frac{a}{c} \qquad (15)\\ &= \lim_{n\to\infty} \frac{-\alpha\beta+q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta+q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta+q_{n-1}}{\alpha+\beta+p_{n-1}} + \frac{-\alpha\beta+q_n}{\alpha+p_n};\\ h(0) &= \frac{b}{d} \qquad (16)\\ &= \lim_{n\to\infty} \frac{-\alpha\beta+q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta+q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta+q_{n-1}}{\alpha+\beta+p_{n-1}} + \frac{-\alpha\beta+q_n}{\beta+p_n}; \end{aligned}$$

and for $k \in \mathbb{Z}$, we have

$$\begin{aligned} h(\lambda^{k+1}) &= \frac{a\lambda^{k+1}+b}{c\lambda^{k+1}+d}\\ &= \lim_{n\to\infty} \frac{-\alpha\beta+q_1}{\alpha+\beta+p_1} + \frac{-\alpha\beta+q_2}{\alpha+\beta+p_2} + \cdots + \frac{-\alpha\beta+q_n}{\alpha+\beta+p_n+\omega_{n-k}}, \qquad (17) \end{aligned}$$

where

$$\omega_n = -\frac{\alpha^n - \beta^n}{\alpha^{n-1} - \beta^{n-1}} \in \widehat{\mathbb{C}}, \qquad n \in \mathbb{Z}.$$

As a first application, we can get quite precise information about the divergence behavior of limit-1 periodic continued fractions of elliptic type (see [6] for more on limit-1 periodic continued fractions of elliptic type).

We consider the case where the continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{18}$$

is a limit 1-periodic continued fraction of elliptic type and, in addition,

$$\sum_{n\geq 1} |a_n - a| < \infty, \qquad \sum_{n\geq 1} |b_n - b| < \infty,$$

for some $a, b \in \mathbb{C}$.

Set

$$d := \left|\frac{b+\sqrt{b^2+4a}}{2}\right| = \left|\frac{b-\sqrt{b^2+4a}}{2}\right|,$$

and define

$$\alpha = \frac{b+\sqrt{b^2+4a}}{2d}, \qquad \beta = \frac{b-\sqrt{b^2+4a}}{2d}.$$

Then $\alpha \neq \beta$, $|\alpha| = |\beta| = 1$. Define, for $n \geq 1$, p_n and q_n by

$$a_n = a + p_n, \qquad b_n = b + q_n.$$

Thus

$$K_{n=1}^{\infty}\frac{a+q_n}{b+p_n}=d\,K_{n=1}^{\infty}\frac{-\alpha\beta+q_n/d^2}{\alpha+\beta+p_n/d}.$$

The second continued fraction satisfies the conditions of Theorem 4.1. Thus this theorem can be applied to all limit 1-periodic continued fractions of elliptic type with $\lim_{n\to\infty}a_n=a$ and $\lim_{n\to\infty}b_n=b$, providing $\sum_{n\geq 1}|a_n-a|<\infty$ and $\sum_{n\geq 1}|b_n-b|<\infty$. Of course, it is known that without any restrictions on how the limit periodic sequences tend to their limits, the behavior can be quit complicated, see [6].

Next, we can obtain (up to a factor of ± 1) the numbers a, b, c, and d in terms of the modified continued fractions and the product for $\det(h)$ given in Theorem 4.1.

Corollary 4.2. *The linear fractional transformation $h(z)$ defined in Theorem 4.1 has the following expression*

$$h(z)=\frac{A(C-B)z+B(A-C)}{(C-B)z+A-C},$$

where $A=h(\infty)$, $B=h(0)$, and $C=h(1)$. Moreover, the constants a, b, c, and d in the theorem have the following formulas

$$a=sA(C-B),\quad b=sB(A-C),\quad c=s(C-B),\quad d=s(A-C),$$

where

$$s=\pm\sqrt{\frac{(\beta-\alpha)\prod_{n=1}^{\infty}\left(1-\frac{q_n}{\alpha\beta}\right)}{(A-B)(C-A)(B-C)}}.$$

It is interesting that the linear fractional transformation which describes the limit set of the divergent continued fraction

$$K_{n=1}^{\infty}\frac{-\alpha\beta+q_n}{\alpha+\beta+p_n}$$

can be described completely in terms of three convergent modified continued fractions.

Let $\mathbb{T}'$ denote the image of $\mathbb{T}$ under h, that is, the limit set of the sequence $\{f_n\}$. The main conclusion of the theorem can be expressed by the statement

$$f_n\sim h(\lambda^{n+1}),\tag{19}$$

where h is the linear fractional transformation in the theorem. It is well known that when λ is not a root of unity, λ^{n+1} is uniformly distributed on $\mathbb{T}$. However, the linear fractional transformation h stretches and compresses arcs of the circle $\mathbb{T}$, so that the distribution of $h(\lambda^{n+1})$ in arcs of $\mathbb{T}'$ is no longer uniform. Thus, although the limit set in the case where λ is not a root of unity is a circle, the concentration of approximants is not uniform around the circle.

Fortunately, the distribution of approximants is completely controlled by the known parameters a, b, c, and d. The following corollary gives the points on the limit sets whose neighborhood arcs have the greatest and least concentrations of approximants. (We do not take the space here to give a precise definition of what this means; interested readers should consult the author's paper [3].)

Corollary 4.3. *When $m=\infty$ and $cd\neq 0$, the points on*

$$\frac{a\mathbb{T}_m+b}{c\mathbb{T}_m+d}$$

with the highest and lowest concentrations of approximants are

$$\frac{\frac{a}{c}|c|+\frac{b}{d}|d|}{|c|+|d|}\qquad \text{and} \qquad \frac{-\frac{a}{c}|c|+\frac{b}{d}|d|}{-|c|+|d|},$$

respectively. If either $c=0$ or $d=0$, then all points on the limit set have the same concentration. The radius of the limit set circle in $\mathbb{C}$ is

$$\left|\frac{\alpha-\beta}{|c|^2-|d|^2}\prod_{n=1}^{\infty}\left(1-\frac{q_n}{\alpha\beta}\right)\right|.$$

The limit set is a line in $\mathbb{C}$ if and only if $|c|=|d|$, and in this case the point of least concentration is ∞.

Corollary 4.4. *If the limit set of the continued fraction in* (12) *is a line in $\mathbb{C}$, then the point of highest concentration of approximants in the limit set is exactly*

$$\frac{h(\infty)+h(0)}{2},$$

the average of the first two modifications of (12) *given in Theorem 4.1.*

It is also possible to derive a convergent continued fractions which have the same limit as the modified continued fractions in Theorem 4.1. These are given in the following corollary.

Corollary 4.5. *Let α, β, $\{p_n\}$, $\{q_n\}$, $h(z)$, be as in Theorem 4.1. Then*

$$\begin{aligned}h(\infty)=-\beta+\frac{q_1+\beta p_1}{\alpha+p_1}&+\frac{(q_1-\alpha\beta)(q_2+\beta p_2)}{(\alpha+p_2)(q_1+\beta p_1)+\beta(q_2+\beta p_2)}\\&+K_{n=3}^{\infty}\frac{(q_{n-1}-\alpha\beta)(q_n+\beta p_n)(q_{n-1}+\beta p_{n-1})}{(\alpha+p_n)(q_{n-1}+\beta p_{n-1})+\beta(q_n+\beta p_n)},\end{aligned}\tag{20}$$

$$\begin{aligned}h(0)=-\alpha+\frac{q_1+\alpha p_1}{\beta+p_1}&+\frac{(q_1-\alpha\beta)(q_2+\alpha p_2)}{(\beta+p_2)(q_1+\alpha p_1)+\alpha(q_2+\alpha p_2)}\\&+K_{n=3}^{\infty}\frac{(q_{n-1}-\alpha\beta)(q_n+\alpha p_n)(q_{n-1}+\alpha p_{n-1})}{(\beta+p_n)(q_{n-1}+\alpha p_{n-1})+\alpha(q_n+\alpha p_n)}.\end{aligned}\tag{21}$$

Let $k\in\mathbb{Z}$ and assume that α/β is not a root of unity. Set

$$\omega_n=-\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha^{n-k-1}-\beta^{n-k-1}},\qquad \text{for } n\geq k':=\max\{3,k+3\}.$$

Then

$$\begin{aligned}h(\lambda^{k+1})=\frac{-\alpha\beta+q_1}{\alpha+\beta+p_1}+\cdots+\frac{-\alpha\beta+q_{k'-1}}{\alpha+\beta+p_{k'-1}}+\frac{-\alpha\beta+q_{k'}}{\alpha+\beta+p_{k'}+\omega_{k'}}\\+\frac{-\alpha\beta+q_{k'+1}-\omega_{k'}(\alpha+\beta+p_{k'+1}+\omega_{k'+1})}{\alpha+\beta+p_{k'+1}+\omega_{k'+1}}+K_{n=k'+2}^{\infty}\frac{c_n}{d_n},\end{aligned}\tag{22}$$

where

$$c_n = (q_{n-1} - \alpha\beta)\,\frac{-\alpha\beta + q_n - \omega_{n-1}\,(\alpha + \beta + p_n + \omega_n)}{-\alpha\beta + q_{n-1} - \omega_{n-2}\,(\alpha + \beta + p_{n-1} + \omega_{n-1})}$$
$$d_n = \alpha + \beta + p_n + \omega_n - \omega_{n-2}\frac{-\alpha\beta + q_n - \omega_{n-1}\,(\alpha + \beta + p_n + \omega_n)}{-\alpha\beta + q_{n-1} - \omega_{n-2}\,(\alpha + \beta + p_{n-1} + \omega_{n-1})}.$$

Before continuing, we give an example which illustrates some of the results mentioned above. Let $|p|$, $|q| < 1$, and define

$$G(p, q, \alpha, \beta) := \frac{-\alpha\beta + q}{\alpha + \beta + p} + \frac{-\alpha\beta + q^2}{\alpha + \beta + p^2} + \cdots + \frac{-\alpha\beta + q^n}{\alpha + \beta + p^n} + \cdots.$$

For $p = 0.3$, $q = 0.2$, $\alpha = \exp(\imath\sqrt{11})$,$\beta = \exp(\imath\sqrt{13})$, we use Corollary 4.5 with $p_n = p^n$, $q_n = q^n$ and $k = -1$ and compute the limits of the three continued fractions there to find

$$\begin{aligned} h(\infty) &= 1.13121 + 0.772998i, \\ h(0) &= 1.20138 + 0.0347473i, \\ h(1) &= -0.412160 - 0.486753i. \end{aligned}$$

We then apply Corollary 4.2 and compute

$$\begin{aligned} s &= 2.97370 + 0.773678i, \\ a &= 0.581867 + 0.408182i, \\ b &= -0.670885 - 0.294104i, \\ c &= 0.518727 + 0.00637067i, \\ d &= -0.565036 - 0.228462i. \end{aligned}$$

With

$$h(z) := \frac{az + b}{cz + d},$$

we now compare the predicted limit set $h(\mathbb{T})$ with the sequence of approximants. Figure 1 shows the first 3000 approximants of $G(0.3, 0.2, \exp(\imath\sqrt{11}), \exp(\imath\sqrt{13}))$ and the circle $h(\mathbb{T})$. We see that the limit set is exactly what is predicted by Theorem 4.1. The large dots show the points of highest (top) and lowest (bottom) points of concentration of approximants, as predicted by Corollary 4.3, namely $1.16911 + 0.374194i$ and $1.60256 - 4.18725i$. We see that prediction and mathematical fact agree in this case also.

We next consider an example where α/β is a root of unity, so that the limit set is finite. We proceed as above to compute $h(z)$ (details omitted). Figure 2 shows the first 3000 approximants of $G(0.3, 0.2, \exp(\imath\sqrt{11}), \exp(\imath(\sqrt{11} + 2\pi/17)))$ and its convergence to the 17 limit points, together with the circle $h(\mathbb{T})$.

Figure 3 shows the image of all seventeen 17th roots of unity under h. Once again the actual limit set and the predicted limit set agree perfectly.

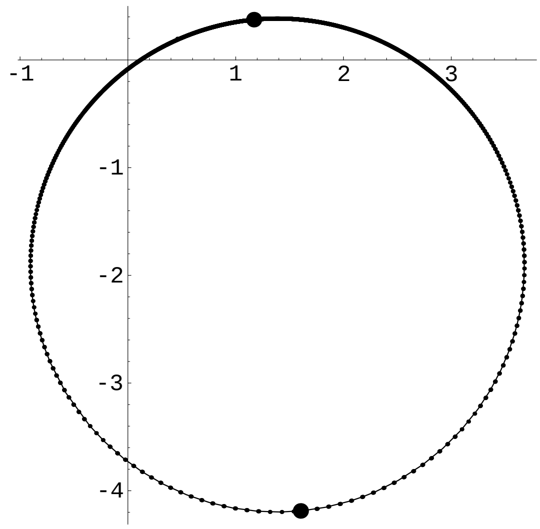

FIGURE 1. The convergence of $G(0.3, 0.2, \exp(\imath\sqrt{11}), \exp(\imath\sqrt{13}))$.

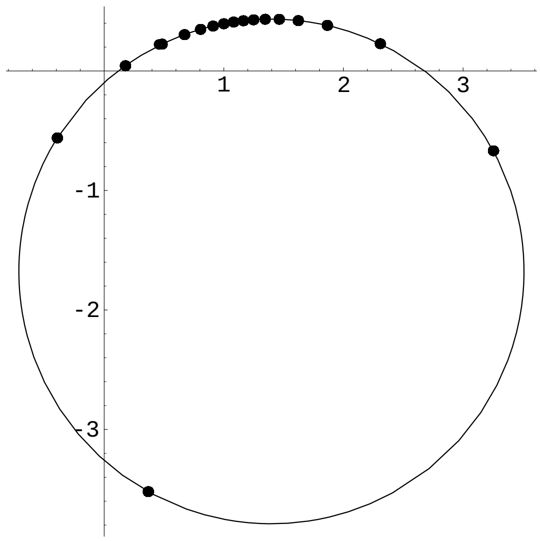

FIGURE 2. The convergence of $G(0.3, 0.2, \exp(\imath\sqrt{11}), \exp(\imath(\sqrt{11} + 2\pi/17)))$.

Lastly, we consider the continued fraction from the beginning of the paper, $K_{n=1}^{\infty} -1/(4/3)$. If we follow the same kind of analysis as above, we find that

$$\begin{aligned} a &= -2/3 + \sqrt{5}/3i, \\ b &= 2/3 + \sqrt{5}/3i, \\ c &= 1, \\ d &= -1. \end{aligned}$$

Corollary 4.4 predicts that the highest concentration of approximants occurs at $(a/c + b/d)/2 = -2/3$. Figure 4 shows the distribution of the first 1200 approximants of the

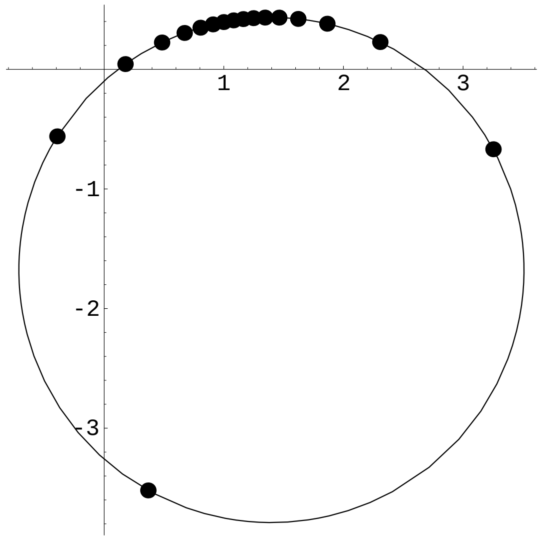

FIGURE 3. The image of the seventeen 17th roots of unity under h.

continued fraction (about 100 extreme values were omitted), once again showing agreement with the theory.

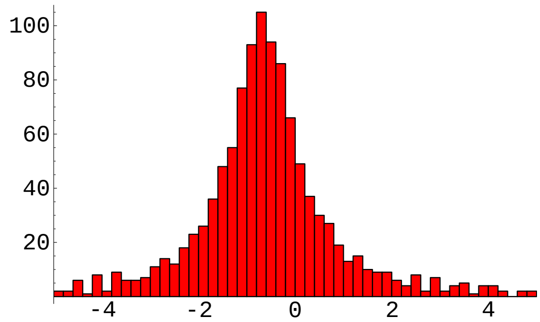

FIGURE 4. The distribution of the first 1200 approximants of $K_{n=1}^{\infty} -1/(4/3)$.

4.1. **An Infinite Family of Divergence Theorems.** An interesting special case of Theorem 4.1 occurs when α and β are distinct m-th roots of unity ($m \geq 2$). In this situation the continued fraction

$$\frac{-\alpha\beta + q_1}{\alpha + \beta + p_1} + \frac{-\alpha\beta + q_2}{\alpha + \beta + p_2} + \frac{-\alpha\beta + q_3}{\alpha + \beta + p_3} + \frac{-\alpha\beta + q_4}{\alpha + \beta + p_4} + \cdots$$

becomes limit periodic and the sequences of approximants in the m different arithmetic progressions modulo m converge. The corollary below, which is also proved in [2], is an easy consequence of Theorem 4.1.

Corollary 4.6. *Let $\{p_n\}_{n\geq 1}$, $\{q_n\}_{n\geq 1}$ be complex sequences satisfying*

$$\sum_{n=1}^{\infty} |p_n| < \infty, \qquad \sum_{n=1}^{\infty} |q_n| < \infty.$$

Let α and β be distinct roots of unity and let m be the least positive integer such that $\alpha^m = \beta^m = 1$. Define

$$G := \frac{-\alpha\beta + q_1}{\alpha + \beta + p_1} + \frac{-\alpha\beta + q_2}{\alpha + \beta + p_2} + \frac{-\alpha\beta + q_3}{\alpha + \beta + p_3} + \cdots.$$

Let $\{P_n/Q_n\}_{n=1}^{\infty}$ denote the sequence of approximants of G. If $q_n \neq \alpha\beta$ for any $n \geq 1$, then G does not converge. However, the sequences of numerators and denominators in each of the m arithmetic progressions modulo m do converge. More precisely, there exist complex numbers $A_0, \dots, A_{m-1}$ and $B_0, \dots, B_{m-1}$ such that, for $0 \leq i < m$,

$$\lim_{k\to\infty} P_{m\,k+i} = A_i, \qquad \lim_{k\to\infty} Q_{m\,k+i} = B_i. \tag{23}$$

Extend the sequences $\{A_i\}$ and $\{B_i\}$ over all integers by making them periodic modulo m so that (23) continues to hold. Then for integers i,

$$A_i = \left(\frac{A_1 - \beta A_0}{\alpha - \beta}\right)\alpha^i + \left(\frac{\alpha A_0 - A_1}{\alpha - \beta}\right)\beta^i, \tag{24}$$

and

$$B_i = \left(\frac{B_1 - \beta B_0}{\alpha - \beta}\right)\alpha^i + \left(\frac{\alpha B_0 - B_1}{\alpha - \beta}\right)\beta^i. \tag{25}$$

Moreover,

$$A_i B_j - A_j B_i = -(\alpha\beta)^{j+1}\frac{\alpha^{i-j} - \beta^{i-j}}{\alpha - \beta}\prod_{n=1}^{\infty}\left(1 - \frac{q_n}{\alpha\beta}\right). \tag{26}$$

Put $\alpha := \exp(2\pi i a/m)$, $\beta := \exp(2\pi i b/m)$, $0 \leq a < b < m$, and $r := m/\gcd(b-a, m)$. Then G has r distinct limits in $\widehat{\mathbb{C}}$ which are given by A_j/B_j, $1 \leq j \leq r$. Finally, for $k \geq 0$ and $1 \leq j \leq r$,

$$\frac{A_{j+kr}}{B_{j+kr}} = \frac{A_j}{B_j}.$$

The number r occuring in this theorem is just the number of distinct limits to which the continued fraction tends. For this reason, we term it the *rank* of the continued fraction.

It is easy to derive general divergence results from this theorem, including Theorem 1.1, the classical Stern-Stolz theorem [6]. The proof of Theorem 1.1 is immediate from Theorem 4.1. Just set $\omega_1 = 1$, $\omega_2 = -1$ (so $m = 2$), $q_n = 0$ and $p_n = b_n$. In fact, Stern-Stolz can be seen as the beginning of an infinite family of divergence theorems. We first give a generalization of Stern-Stolz, then give a corollary describing the infinite family. Last, we list the first few examples in the infinite family.

To obtain the generalization, take $q_n = a_n$ instead of $q_n = 0$.

Corollary 4.7. *Let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \neq -1$ for $n \geq 1$, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$. Then*

$$b_0 + K_{n=1}^{\infty}\frac{1 + a_n}{b_n}$$

diverges. In fact, for $p = 0, 1$,

$$\lim_{n\to\infty} P_{2n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{2n+p} = B_p \neq \infty,$$

and

$$A_1B_0 - A_0B_1 = \prod_{n=1}^{\infty}(1+a_n).$$

Proof. This follows immediately from Theorem 4.1, upon setting $\omega_1 = 1$, $\omega_2 = -1$ (so $m = 2$), $q_n = a_n$ and $p_n = b_n$. □

We have not been able to find Corollary 4.7 in the literature.

The natural infinite family of Stern-Stolz type theorems is described by the following corollary.

Corollary 4.8. *Let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \neq 1$ for $n \geq 1$, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$. Let $m \geq 3$ and let ω_1 be a primitive m-th root of unity. Then*

$$b_0 + K_{n=1}^{\infty}\frac{-1+a_n}{\omega_1+\omega_1^{-1}+b_n}$$

does not converge, but the numerator and denominator convergents in each of the m arithmetic progressions modulo m do converge. If m is even, then for $1 \leq p \leq m/2$,

$$\begin{aligned}\lim_{n\to\infty} P_{mn+p} &= -\lim_{n\to\infty} P_{mn+p+m/2} = A_p \neq \infty,\\ \lim_{n\to\infty} Q_{mn+p} &= -\lim_{n\to\infty} Q_{mn+p+m/2} = B_p \neq \infty.\end{aligned}$$

If m is odd, then the continued fraction has rank m. If m is even, then the continued fraction has rank $m/2$. Further, for $2 \leq p \leq m'$, where $m' = m$ if m is odd and $m/2$ if m is even,

$$A_pB_{p-1} - A_{p-1}B_p = -\prod_{n=1}^{\infty}(1-a_n).$$

Proof. In Theorem 4.1, let $\omega_2 = 1/\omega_1$. □

Some explicit examples are given below.

Example 1. *Let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \neq 1$ for $n \geq 1$, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$. Then each of the following continued fractions diverges:*

(i) The following continued fraction has rank three:

$$b_0 + K_{n=1}^{\infty}\frac{-1+a_n}{1+b_n}. \tag{27}$$

In fact, for $p = 1, 2, 3$,

$$\begin{aligned}\lim_{n\to\infty} P_{6n+p} &= -\lim_{n\to\infty} P_{6n+p+3} = A_p \neq \infty,\\ \lim_{n\to\infty} Q_{6n+p} &= -\lim_{n\to\infty} Q_{6n+p+3} = B_p \neq \infty.\end{aligned}$$

(ii) The following continued fraction has rank four:

$$b_0 + K_{n=1}^{\infty}\frac{-1+a_n}{\sqrt{2}+b_n}. \tag{28}$$

In fact, for $p = 1, 2, 3, 4,$

$$\lim_{n\to\infty} P_{8n+p} = -\lim_{n\to\infty} P_{8n+p+4} = A_p \neq \infty,$$
$$\lim_{n\to\infty} Q_{8n+p} = -\lim_{n\to\infty} Q_{8n+p+4} = B_p \neq \infty.$$

(iii) The following continued fraction has rank five:

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{(1-\sqrt{5})/2 + b_n}. \tag{29}$$

In fact, for $p = 1, 2, 3, 4, 5,$

$$\lim_{n\to\infty} P_{5n+p} = A_p \neq \infty, \qquad \lim_{n\to\infty} Q_{5n+p} = B_p \neq \infty.$$

(iv) The following continued fraction has rank six:

$$b_0 + K_{n=1}^{\infty} \frac{-1 + a_n}{\sqrt{3} + b_n}. \tag{30}$$

In fact, for $p = 1, 2, 3, 4, 5, 6,$

$$\lim_{n\to\infty} P_{12n+p} = -\lim_{n\to\infty} P_{12n+p+6} = A_p \neq \infty,$$
$$\lim_{n\to\infty} Q_{12n+p} = -\lim_{n\to\infty} Q_{12n+p+6} = B_p \neq \infty.$$

In each case we have, for p *in the appropriate range, that*

$$A_p B_{p-1} - A_{p-1} B_p = -\prod_{n=1}^{\infty} (1 - a_n).$$

Proof. In Corollary 4.8, set
(i) $\omega_1 = \exp(2\pi i/6)$;
(ii) $\omega_1 = \exp(2\pi i/8)$;
(iii) $\omega_1 = \exp(2\pi i/5)$; (iv) $\omega_1 = \exp(2\pi i/12)$. □

The cases $\omega_1 = \exp(2\pi i/m)$, $m = 3, 4, 10$ give continued fractions that are the same as those above after an equivalence transformation and renormalization of the sequences $\{a_n\}$ and$\{b_n\}$. Note that the continued fractions (28) and (30) are, after an equivalence transformation and renormalizing the sequences $\{a_n\}$ and $\{b_n\}$, of the forms

$$b_0 + K_{n=1}^{\infty} \frac{-2 + a_n}{2 + b_n}, \tag{31}$$

and

$$b_0 + K_{n=1}^{\infty} \frac{-3 + a_n}{3 + b_n}, \tag{32}$$

respectively. Because of the equivalence transformations employed, the convergents do not tend to limits in (31) or (32). Also, it should be mentioned that Theorem 3.3 of [1] is essentially the special case $a_n = 0$ of part (i) of our example. Nevertheless (31) and (32) have ranks 4 and 6 respectively.

Corollary 4.6 now makes it trivial to construct q-continued fractions with arbitrarily many limits.

Example 2. *Let $f(x)$, $g(x) \in \mathbb{Z}[q][x]$ be polynomials with zero constant term. Let ω_1, ω_2 be distinct roots of unity and suppose m is the least positive integer such that $\omega_1^m = \omega_2^m = 1$. Define*

$$G(q) := \frac{-\omega_1\omega_2 + g(q)}{\omega_1 + \omega_2 + f(q)} + \frac{-\omega_1\omega_2 + g(q^2)}{\omega_1 + \omega_2 + f(q^2)} + \frac{-\omega_1\omega_2 + g(q^3)}{\omega_1 + \omega_2 + f(q^3)} + \cdots .$$

Let $|q| < 1$. If $g(q^n) \neq \omega_1\omega_2$ for any $n \geq 1$, then $G(q)$ does not converge. However, the sequences of approximants of $G(q)$ in each of the m arithmetic progressions modulo m converge to values in $\hat{\mathbb{C}}$. The continued fraction has rank $m/\gcd(b-a,m)$, where a and b are as defined in Theorem 4.1.

From this example we can conclude that (2) and (3) are far from unique examples and many other q-continued fractions with multiple limits can be immediately written down. Thus, to Ramanujanize a bit, one can immediately see that the continued fractions

$$\operatorname*{K}_{n\geq 1}^{\infty} \frac{-1/2}{1+q^n} \qquad \text{and} \qquad \operatorname*{K}_{n\geq 1}^{\infty} \frac{-1/2+q^n}{1+q^n} \tag{33}$$

both have rank four, while the continued fractions

$$\operatorname*{K}_{n\geq 1}^{\infty} \frac{-1/3}{1+q^n} \qquad \text{and} \qquad \operatorname*{K}_{n\geq 1}^{\infty} \frac{-1/3+q^n}{1+q^n} \tag{34}$$

both have rank six.

4.2. **Application: Generalization of a Continued Fraction of Ramanujan.** In [3] we gave a non-trivial example of the preceding theory, the inspiration for which is a beautiful result of Ramanujan.

Theorem 4.9. *Let $|q| < 1$, $|\alpha| = |\beta| = 1$, $\alpha \neq \beta$, and the order of $\lambda := \alpha/\beta$ in $\mathbb{T}$ be m. For x, $y \neq 0$ and fixed $|q| < 1$, define*

$$P(x,y) = \sum_{n=0}^{\infty} \frac{x^n q^{n(n+1)/2}}{(q)_n (y\,q)_n}.$$

Then

$$\begin{aligned} l.s.\left(\frac{-\alpha\beta}{\alpha+\beta+q} - \frac{\alpha\beta}{\alpha+\beta+q^2} - \frac{\alpha\beta}{\alpha+\beta+q^3 \cdots}\right) \\ = -\frac{\beta P(q\alpha^{-1}, \beta\alpha^{-1})\mathbb{T}_m - \alpha P(q\beta^{-1}, \alpha\beta^{-1})}{P(\alpha^{-1}, \beta\alpha^{-1})\mathbb{T}_m - P(\beta^{-1}, \alpha\beta^{-1})}. \end{aligned} \tag{35}$$

Moreover,

$$\begin{aligned} \frac{-\alpha\beta}{\alpha+\beta+q} - \frac{\alpha\beta}{\alpha+\beta+q^2} - \frac{\alpha\beta}{\alpha+\beta+q^3 \cdots} - \frac{\alpha\beta}{\alpha+\beta+q^n} \\ \sim -\frac{\beta P(q\alpha^{-1}, \beta\alpha^{-1})\lambda^{n+1} - \alpha P(q\beta^{-1}, \alpha\beta^{-1})}{P(\alpha^{-1}, \beta\alpha^{-1})\lambda^{n+1} - P(\beta^{-1}, \alpha\beta^{-1})}. \end{aligned} \tag{36}$$

In [3] we also used the Bauer-Muir Transform to produce some convergent continued fractions. One such example is the following.

Corollary 1. *Let $|q|<1$ and let α and β be distinct points on the unit circle such that α/β is not a root of unity. Then*

$$-\beta+\frac{\beta q}{\alpha+q}+K_{n=2}^{\infty}\frac{-\alpha\beta q}{q^n+\alpha+\beta q}=-\beta\frac{\displaystyle\sum_{n=0}^{\infty}\frac{\alpha^{-n}q^{n(n+3)/2}}{(q;q)_n(\beta q/\alpha;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{\alpha^{-n}q^{n(n+1)/2}}{(q;q)_n(\beta q/\alpha;q)_n}}. \tag{37}$$

5. Poincaré type recurrences

Let the sequence $\{x_n\}_{n\geq 0}$ have the initial values $x_0, \ldots, x_{p-1}$ and be subsequently defined by

$$x_{n+p}=\sum_{r=0}^{p-1}a_{n,r}x_{n+r}, \tag{38}$$

for $n\geq 0$. Suppose also that there are numbers $a_0,\ldots,a_{p-1}$ such that

$$\lim_{n\to\infty}a_{n,r}=a_r, \qquad 0\leq r\leq p-1. \tag{39}$$

A recurrence of the form (38) satisfying the condition (39) is called a Poincaré-type recurrence, (39) being known as the Poincaré condition. Such recurrences were initially studied by Poincaré who proved that if the roots of the characteristic equation

$$t^p-a_{p-1}t^{p-1}-a_{p-2}t^{p-2}-\cdots-a_0=0 \tag{40}$$

have distinct norms, then the ratios of consecutive terms in the recurrence (for any set of initial conditions) tend to one of the roots. See [8]. Because the roots are also the eigenvalues of the associated companion matrix, they are also referred to as the eigenvalues of (38). This result was improved by O. Perron, who obtained a number of theorems about the limiting asymptotics of such recurrence sequences. Perron [7] made a significant advance in 1921 when he proved the following theorem which for the first time treated cases of eigenvalues which repeat or are of equal norm.

Theorem 5.1. *Let the sequence $\{x_n\}_{n\geq 0}$ be defined by initial values $x_0, \ldots, x_{p-1}$ and by (38) for $n\geq 0$. Suppose also that there are numbers $a_0,\ldots,a_{p-1}$ satisfying (39). Let q_1, $q_2, \ldots q_\sigma$ be the distinct moduli of the roots of the characteristic equation (40) and let l_λ be the number of roots whose modulus is q_λ, multiple roots counted according to multiplicity, so that*

$$l_1+l_2+\ldots l_\sigma=p.$$

Then, provided $a_{n,0}$ be different from zero for $n\geq 0$, the difference equation (38) has a fundamental system of solutions, which fall into σ classes, such that, for the solutions of the λ-th class and their linear combinations,

$$\limsup_{n\to\infty}\sqrt[n]{|x_n|}=q_\lambda.$$

The number of solutions of the λ-th class is l_λ.

Thus when all of the characteristic roots have norm 1, this theorem gives that

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = 1.$$

Another related paper is [4] where the authors study products of matrices and give a sufficient condition for their boundedness. This is then used to study "equimodular" limit periodic continued fractions, which are limit periodic continued fractions in which the characteristic roots of the associated 2×2 matrices are all equal in modulus. The matrix theorem in [4] can also be used to obtain results about the boundedness of recurrence sequences. We study a more specialized situation here and obtain far more detailed information as a consequence.

Our focus is on the case where the characteristic roots are distinct numbers on the unit circle. Under a condition stronger than (39) we have a theorem showing that all non-trivial solutions of such recurrences approach a limit set in a precisely controlled way. Specifically, our theorem is:

Theorem 5.2. *Let the sequence $\{x_n\}_{n\geq 0}$ be defined by initial values $x_0, \dots, x_{p-1}$ and by*

$$x_{n+p} = \sum_{r=0}^{p-1} a_{n,r} x_{n+r}, \tag{41}$$

for $n \geq 0$. Suppose also that there are numbers $a_0, \dots, a_{p-1}$ such that

$$\sum_{n=0}^{\infty} |a_r - a_{n,r}| < \infty, \qquad 0 \leq r \leq p-1.$$

Suppose further that the roots of the characteristic equation

$$t^p - a_{p-1}t^{p-1} - a_{p-2}t^{p-2} - \cdots - a_0 = 0 \tag{42}$$

are distinct and all on the unit circle, with values, say, $\alpha_0, \dots, \alpha_{p-1}$. Then there exist complex numbers $c_0, \dots, c_{p-1}$ such that

$$x_n \sim \sum_{i=0}^{p-1} c_i \alpha_i^n. \tag{43}$$

The following corollary, also proved in [2], is immediate.

Corollary 5.3. *Let the sequence $\{x_n\}_{n\geq 0}$ be defined by initial values $x_0, \dots, x_{p-1}$ and by*

$$x_{n+p} = \sum_{r=0}^{p-1} a_{n,r} x_{n+r},$$

for $n \geq 0$. Suppose also that there are numbers $a_0, \dots, a_{p-1}$ such that

$$\sum_{n=0}^{\infty} |a_r - a_{n,r}| < \infty, \qquad 0 \leq r \leq p-1.$$

Suppose further that the roots of the characteristic equation

$$t^p - a_{p-1}t^{p-1} - a_{p-2}t^{p-2} - \cdots - a_0 = 0$$

are distinct roots of unity, say $\alpha_0, \ldots, \alpha_{p-1}$*. Let* m *be the least positive integer such that, for all* $j \in \{0, 1, \ldots, p-1\}$, $\alpha_j^m = 1$*. Then, for* $0 \leq j \leq m-1$*, the subsequence* $\{x_{mn+j}\}_{n=0}^{\infty}$ *converges. Set* $l_j = \lim_{n\to\infty} x_{nm+j}$*, for integers* $j \geq 0$*. Then the (periodic) sequence* $\{l_j\}$ *satisfies the recurrence relation*

$$l_{n+p} = \sum_{r=0}^{p-1} a_r l_{n+r},$$

and thus there exist constants $c_0, \cdots, c_{p-1}$ *such that*

$$l_n = \sum_{i=0}^{p-1} c_i \alpha_i^n.$$

6. Applications to (r, s)-matrix continued fractions

In [5], the authors define a generalization of continued fractions called (r, s)-matrix continued fractions. This generalization unifies a number of generalizations of continued fractions including "generalized (vector valued) continued fractions" and "G-continued fractions", see [6] for terminology.

Here we show that our results apply to limit periodic (r, s)-matrix continued fractions with eigenvalues of equal magnitude, giving estimates for the asymptotics of their approximants so that their limit sets can be determined.

For consistency we closely follow the notation used in [5] to define (r, s)-matrix continued fractions. Let $M_{s,r}(\mathbb{C})$ denote the set of $s \times r$ matrices over the complex numbers. Let θ_k be a sequence of $n \times n$ matrices over $\mathbb{C}$. Assume that $r + s = n$. A (r, s)-matrix continued fraction is associated with a recurrence system of the form $Y_k = Y_{k-1}\theta_k$. The continued fraction is defined by its sequence of approximants. These are sequences of $s \times r$ matrices defined in the following manner.

Define the function $f : D \in M_n(\mathbb{C}) \to M_{s,r}(\mathbb{C})$ by

$$f(D) = B^{-1}A, \tag{44}$$

where B is the $s \times s$ submatrix consisting of the last s elements from both the rows and columns of D, and A is the $s \times r$ submatrix consisting of the first r elements from the last s rows of D.

Then the k-th approximant of the (r, s)-matrix continued fraction associated with the sequence θ_k is defined to be

$$s_k := f(\theta_k \theta_{k-1} \cdots \theta_2 \theta_1). \tag{45}$$

To apply Theorem 1 to this situation, we endow $M_{s\times r}(\mathbb{C})$ with a metric by letting the distance function for two such matrices be the maximum absolute value of the respective differences of corresponding pairs of elements. Then, providing that the f is continuous, (a suitable specialization of) our theorem can be applied. (Note that f will be continuous providing that it exists, since the inverse function of a matrix is continuous when it exists.)

Let $\lim_{k\to\infty} \theta_k = \theta$, for some $\theta \in M_n(\mathbb{C})$. Then the recurrence system is said to be of Poincaré type and the (r, s)-matrix continued fraction is said to be limit periodic. Under our usual condition, Theorem 1 can be applied and the following theorem results.

Theorem 6.1. *Suppose that the condition $\sum_{k\geq 1}||\theta_k-\theta||<\infty$ holds, that the matrix θ is diagonalizable, and that the eigenvalues of θ are all of magnitude 1. Then the kth approximant s_k has the asymptotic formula*

$$s_k \sim f(\theta^k F), \tag{46}$$

where F is the matrix defined by the convergent product

$$F := \lim_{k\to\infty} \theta^{-k}\theta_k\theta_{k-1}\cdots\theta_2\theta_1.$$

Note that because of the way that (r,s)-matrix continued fractions are defined, we have taken products in the reverse order than the rest of the paper.

As a consequence of this asymptotic, the limit set can be determined from

$$l.s.(s_k) = l.s.(f(\theta^k F)).$$

7. Conclusion

Because of length restrictions, we have omitted several corollaries as well as most proofs. Interested readers should consult the author's papers [2] and [3].

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