THE p-DISSECTION OF A PRODUCT OF QUINTUPLE PRODUCTS

TAYLOR DANIELS, TIM HUBER, JAMES MCLAUGHLIN, AND DONGXI YE

ABSTRACT. Let $p \equiv 1 \pmod 4$ be prime, $p = m^2 + n^2$, and let b be a positive integer. Let $Q(z, q) = (z, q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty$ denote the product appearing in the quintuple product identity. We derive explicit formulae for the p-dissection of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$, and determine sign patterns (mod p) of the Taylor series coefficients of the associated quotient $Q(q^{bm}, q^p)Q(q^{bn}, q^p)/(q^p; q^p)^2_\infty$.

As an example of our results, let $p \equiv 1 \pmod{12}$ such that $3|n$. If $m \equiv 1 \pmod 3$, then

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) = \sum_{k=0}^{p-1} q^{3bkm+p\left(3k^2-k\right)/2} Q(q^{p(6b+(6k-1)m-n+p)/6}, q^{p^2})Q(q^{p(-6kn-m+n+p)/6}, q^{p^2}).$$

There is a similar expansion when $m \equiv -1 \pmod 3$. When $p \equiv 5 \pmod{12}$, the q-products in the p-dissection involve the product that occurs in Winquist's identity.

Some combinatorial applications of the p-dissection formulae are also given.

1. Introduction

Using the standard q-Pochhammer symbols $(a_1, \ldots, a_j; q)_\infty = (a_1; q)_\infty \cdots (a_j; q)_\infty$, let

$$Q(z, q) := (z, q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty \tag{1.1}$$

denote the *quintuple product*. In a recent paper [10], two of the present authors proved the following result: if we define the quantities b_n via

$$B(q) := Q(q^2, q^{13})Q(q^5, q^{13})Q(q^6, q^{13}) =: \sum_{n=0}^{\infty} b_n q^n,$$

then $b_{13t+3} = b_{13t+9} = b_{13t+11} = 0$, for all $t \geq 0$. This led us to ask if there existed other products of quintuple products with coefficients that vanish in arithmetic progressions. This paper explains the vanishing for pairs of quintuple products by addressing the following:

> When do products of the form $Q(q^r, q^p)Q(q^s, q^p)$ have p-dissections explaining instances where coefficients vanish in arithmetic progressions? What further information, such as sign patterns and parity follow from these expansions?

Extensive computation suggested that systematic vanishing occurs when the exponents r, s are linked to a representation of the prime p as the sum of two squares when $p \equiv 1 \pmod 4$. For b, m, n positive integers and $p = m^2 + n^2$, we consider the products

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) =: \sum_{t=0}^{\infty} a_t q^t. \tag{1.2}$$

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Our first main goal is to derive, for the components $\sum_{t\equiv r \pmod p} a_t q^t$, $0 \le r \le p-1$, of the p-dissection, explicit formulae for these components in terms of either products of two quintuple products $Q(-,-)$ at appropriate arguments (when $p \equiv 1 \pmod{12}$), or in terms of the products appearing in Winquist's identity (when $p \equiv 5 \pmod{12}$). We remark that these product representations of the dissection components were first found experimentally by considering the dissection components for $p = 13, 17, 29$ and 37 and finding that these could be written as infinite q-products. We then give general conditions implying that two residue classes modulo p contain only zero coefficients. For instance, when $p \equiv 5 \pmod{12}$, one has

$$a_{pt+bw(1-3b\bar{m})} = a_{pt+bw(1-3b\bar{n})} = 0, \qquad w \equiv \bar{2}(m+n) \pmod p. \tag{1.3}$$

A similar representation exists for the vanishing classes of coefficients when $p \equiv 1 \pmod{12}$.

We also analyze the coefficients of the quotient

$$\frac{Q(q^{bm}, q^p)Q(q^{bn}, q^p)}{(q^p;q^p)_\infty^2}, \tag{1.4}$$

showing that, after ignoring finitely many initial zero terms, the signs within each residue class modulo p follow explicit, predictable patterns.

The dissections in this paper are made possible by the dissection formula of Liu and Yang [15, Thm. 2] for products of pairs of triple products $\langle z;q\rangle_\infty$, where throughout this paper we use the notation

$$\langle z;q\rangle_\infty = (z, q/z, q;q)_\infty \qquad \text{and} \qquad \langle z_1,\ldots,z_k;q\rangle_\infty = \langle z_1;q\rangle_\infty \cdots \langle z_k;q\rangle_\infty.$$

By applying these formulae in Theorem 3 for $\langle -uq;q^2\rangle_\infty \langle -vq;q^2\rangle_\infty$, and using further relations for $\langle z;q\rangle_\infty$, we derive p-dissections for $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$ for any prime $p \equiv 1 \pmod 4$. An example of the dissections from this class appears in Theorem 1, a special case of Theorem 3 addressing $p \equiv 1 \pmod{12}$.

Theorem 1. *Let $p \equiv 1 \pmod{12}$ with $p = m^2 + n^2$ and b be a positive integer as above, with $3|n$. If $m \equiv 1 \pmod 3$, then*

$$\begin{aligned} &Q(q^{bm}, q^p)Q(q^{bn}, q^p) \\ &\quad = \sum_{k=0}^{p-1} q^{3bkm + \frac{k(3k-1)p}{2}} Q(q^{bp+kmp+\frac{(p-m-n)p}{6}}, q^{p^2}) Q(q^{-knp+\frac{(p-m+n)p}{6}}, q^{p^2}). \end{aligned}$$

When $p \equiv 5 \pmod{12}$, the dissections involve a product appearing in Winquist's identity. A typical dissection from the more general Theorem 4 is given in Theorem 2 below.

Theorem 2. *Let $p \equiv 5 \pmod{12}$ with $p = m^2 + n^2$, let b be a positive integer, and let*

$$W(a,b,q) = \langle a, b, ab, \tfrac{a}{b}; q\rangle_\infty (q;q)_\infty^{-2}.$$

If $m + n \equiv 0 \pmod 3$ and

$$\alpha_k := 3kmp + 3bp - \frac{(m+n)p}{2} + \frac{3p^2}{2}, \qquad \beta_k := 3knp + \frac{(m-n)p}{2} + \frac{3p^2}{2},$$

$$\chi_k := 3bkm + p\Big[\tfrac{k(3k-1)}{2} + (1-ns)(kn + \tfrac{(p+m-n(1+ps))}{6})\Big], \tag{1.5}$$

one has

$$Q(q^{bm},q^p)Q(q^{bn},q^p)=\sum_{k=0}^{p-1} q^{\chi_k} W(-q^{\alpha_k/3},-q^{(\beta_k-nsp^2)/3},q^{p^2}). \tag{1.6}$$

The p-dissection we derive lets us recover our vanishing coefficient results like (1.3) fairly easily—see Corollaries 3 and 5. The dissections also allow us to derive explicit, eventually periodic sign patterns for the coefficients of the quotients (1.4), with period p, and tabulate these patterns in representative examples—see Corollaries 4 and 7. Finally, the form of the dissection components allow us to show that certain p-components vanish modulo 2.

We make this preview of the results in the paper a little more concrete by briefly describing some particular examples that are given later in the paper of the general results described above. In Example 1, it is shown for $Q(q^{10},q^{13})Q(q^{15},q^{13})$ that one has (with the notation of (1.2)) $a_{13t+6}=a_{13t+9}=0$. Example 2 shows (apart from some initial zero coefficients) for the same function, that the signs of the sequence of coefficients a_t is periodic modulo 13, and states this sign pattern explicitly. Examples 3 and 5 exhibit similar results for the function $Q(q^2,q^{17})Q(q^8,q^{17}) =: \sum_{t=0}^{\infty} a_t q^2$. Example 4 shows for this same function that $a_{17t+5}\equiv a_{17t+10}\equiv 0 \pmod 2$. Example 6 states explicitly the 13-dissection of $F(q):=Q(q^2,q^{13})Q(q^3,q^{13})$ and hence derives several partition theoretic results from the 13-dissection of $F(q)/(q^{13};q^{13})^2_\infty$. Example 7 similarly states explicitly the 17-dissection of $Q(q^2,q^{17})Q(q^8,q^{17})$ and likewise derives some partition theoretic consequences.

The remainder of the paper is organized as follows: Section 2 records notation and preliminary lemmas, including the including the dissection identities for the triple product. Section 3 proves the main p-dissection formulae for the case $p\equiv 1 \pmod{12}$ and derives vanishing and sign consequences. Section 4 treats the case $p\equiv 5 \pmod{12}$ via Winquist's identity. Section 5 gives combinatorial interpretations for some cases, and Section 6 lists concluding remarks and open problems.

2. Preliminaries

Throughout this paper, p denotes a prime satisfying $p\equiv 1 \pmod 4$. For fixed p, "overbarred" quantities (such as $\bar{m}$) indicate multiplicative inverses $\pmod p$. Expressions of the form $\frac{a}{b} \pmod p$ with $(b,p)=1$ are understood to mean $a\bar{b} \pmod p$. Let

$$\langle z_1,\dots,z_n;q\rangle_\infty := \langle z_1;q\rangle_\infty \cdots \langle z_n;q\rangle_\infty, \qquad \text{with} \qquad \langle z;q\rangle_\infty=(z,q/z,q;q)_\infty.$$

It is straightforward to check that

$$\langle z^{-1};q\rangle_\infty = -z^{-1}\langle z;q\rangle_\infty.$$

We recall the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2} = \langle z;q\rangle_\infty = (z,q/z,q;q)_\infty \qquad \text{for } |q|<1 \text{ and } z\neq 0, \tag{2.1}$$

and recall several equivalent versions of the quintuple product identity, namely

$$\begin{aligned} Q(z,q) = \sum_{n=-\infty}^{\infty} q^{(3n-1)n/2} z^{3n}(1-zq^n) &= \langle z;q\rangle_\infty (qz^2,q/z^2;q^2)_\infty \\ &= \langle -qz^3;q^3\rangle_\infty - z\langle -q^2z^3;q^3\rangle_\infty, \end{aligned} \tag{2.2}$$

again for $|q| < 1$ and $z \neq 0$.

In discussing a q-series, say $\sum_{t\in\mathbb{Z}} a_t q^t$, for $0 \le r \le p-1$ the $r \pmod p$ *component* of the series is $\sum_{t\in\mathbb{Z}} a_{tp+r} q^{tp+r}$. When p is fixed and there is no risk of confusion, we may simply refer to the "r-component" of a series.

It is elementary to verify the (equivalent) equalities

$$\langle -q^x; q^y\rangle_\infty = q^x \langle -q^{x+y}; q^y\rangle_\infty \qquad \text{and} \qquad \langle -q^x; q^y\rangle_\infty = q^{y-x}\langle -q^{x-y}; q^y\rangle_\infty,$$

so that the x in $\langle q^x; q^y\rangle_\infty$ may be shifted up or down by y at a "cost" of q^x or q^{y-x}, respectively. Applying these shifts multiple times, we have the following useful equalities, which we record in a lemma for convenient reference.

Lemma 1. *For all integers t, one has*

$$\langle -q^{x+ty}; q^y\rangle_\infty = q^{(y-x)t+\frac{t(t+1)y}{2}} \langle -q^x; q^y\rangle_\infty, \tag{2.3}$$

$$\langle -q^{x-ty}; q^y\rangle_\infty = q^{tx-\frac{t(t+1)y}{2}} \langle -q^x; q^y\rangle_\infty. \tag{2.3$'$}$$

The following elementary lemma is also quite useful for our arguments.

Lemma 2. *If $y > 0$ is an integer and $x \equiv 0 \pmod y$, then $\langle q^x; q^y\rangle_\infty = 0$.*

Proof. As $\langle q^x; q^y\rangle_\infty = (q^x, q^{y-x}, q^y; q^y)_\infty$ has factors $(1-q^{x+ty})$ and $(1-q^{-x+(t+1)y})$ for all integer $t \ge 0$, if y divides x then, for some t, one of these factors will be $(1-q^0) = 0$. □

The following is a specialization of a general result due to Liu and Yang.

Lemma 3. [15, Thm.2] *Let $p = m^2 + n^2$ with a and b coprime. For $|q| < 1$, one has*

$$\langle -uq; q^2\rangle_\infty \langle -vq; q^2\rangle_\infty = \sum_{k=0}^{p-1} q^{k^2} u^k \langle -u^m v^n q^{2km+p}; q^{2p}\rangle_\infty \langle -u^n v^{-m} q^{2kn+p}; q^{2p}\rangle_\infty. \tag{2.4}$$

We aim to use Lemma 3 to rewrite the products of the form

$$\langle -q^{p\pm 3bm}; q^{3p}\rangle_\infty \langle -q^{p\pm 3bn}; q^{3p}\rangle_\infty$$

that appear in the expansion of

$$\begin{aligned} &Q(q^{bm}, q^p) Q(q^{bn}, q^p) \\ &= \left[\langle -q^{p+3bm}; q^{3p}\rangle_\infty - q^{bm}\langle -q^{p-3bm}; q^{3p}\rangle_\infty\right]\left[\langle -q^{p+3bn}; q^{3p}\rangle_\infty - q^{bn}\langle -q^{p-3bn}; q^{3p}\rangle_\infty\right]. \end{aligned} \tag{2.5}$$

In particular, if $p = m^2 + n^2$, then replacing

$$q \to q^{\frac{3p}{2}}, \quad u \to q^a, \quad \text{and} \quad v \to q^b,$$

equation (2.4) states that

$$\begin{aligned} &\langle -q^{a+\frac{3p}{2}}; q^{3p}\rangle_\infty \langle -q^{b+\frac{3p}{2}}; q^{3p}\rangle_\infty \\ &\qquad = \sum_{k=0}^{p-1} q^{ak+\frac{3k^2p}{2}} \langle -q^{3kmp+(am+bn)+\frac{y}{2}}; q^y\rangle_\infty \langle -q^{3knp+(an-bm)+\frac{y}{2}}; q^y\rangle_\infty. \end{aligned} \tag{2.6}$$

Letting

$$\alpha = 3bm - \frac{p}{2} \qquad \text{and} \qquad \beta = 3bn - \frac{p}{2},$$

using the pair (α,β) for (a,b) in (2.6) evidently produces $\langle -q^{p+3bm};q^{3p}\rangle_\infty\langle -q^{p+3bn};q^{3p}\rangle_\infty$ on the left-hand side there. For the product $\langle -q^{p+3bm};q^{3p}\rangle_\infty\langle -q^{p-3bn};q^{3p}\rangle_\infty$, we use the pair $(\alpha,\beta+p)$; indeed, in this case we have

$$\langle -q^{p-3bn};q^{3p}\rangle_\infty = \langle -q^{3bn+2p};q^{3p}\rangle_\infty = \langle -q^{\beta+p+\frac{3p}{2}};q^{3p}\rangle_\infty.$$

Letting

$$a_0 = 3bm - \frac{p}{2} \qquad \text{and} \qquad a_0 = 3bn - \frac{p}{2},$$

using the pair (a_0,b_0) for (a,b) in (2.6) evidently produces $\langle -q^{p+3bm},-q^{p+3bn};q^{3p}\rangle_\infty$ on the left-hand side there. Continuing along this line, in total we find that the pairs

$$(a_0,b_0), \quad (a_0+p,b_0), \quad (a_0,b_0+p), \quad \text{and} \quad (a_0+p,b_0+p)$$

produce the products

$$\begin{aligned} &\langle -q^{3bm+p},-q^{3bn+p};q^{3p}\rangle_\infty, && \langle -q^{3bm+2p},-q^{3bn+p};q^{3p}\rangle_\infty,\\ &\langle -q^{3bm+p},-q^{3bn+2p};q^{3p}\rangle_\infty, \quad \text{and} && \langle -q^{3bm+2p},-q^{3bn+2p};q^{3p}\rangle_\infty, \end{aligned}$$

respectively.

Definition 1. For all integers k, let

$$\alpha_k := 3bp + 3kmp - \tfrac{(m+n)p}{2} + \tfrac{3p^2}{2}, \tag{2.7a}$$

$$\beta_k := 3knp + \tfrac{(m-n)p}{2} + \tfrac{3p^2}{2}, \tag{2.7b}$$

$$\gamma_k := 3bkm + \tfrac{k(3k-1)p}{2}. \tag{2.7c}$$

The following is an immediate consequence of Lemma 3.

Corollary 1. *Let*

$$y = 3p^2.$$

Then the following expansions hold:

$$\langle -q^{3bm+p},-q^{3bn+p};q^{3p}\rangle_\infty = \sum_{k=0}^{p-1} q^{\gamma_k}\langle -q^{\alpha_k},-q^{\beta_k};q^y\rangle_\infty, \tag{2.8a}$$

$$\langle -q^{3bm+p},-q^{3bn+2p};q^{3p}\rangle_\infty = \sum_{k=0}^{p-1} q^{\gamma_k}\langle -q^{\alpha_k+np},-q^{\beta_k-mp};q^y\rangle_\infty, \tag{2.8b}$$

$$\langle -q^{3bm+2p},-q^{3bn+p};q^{3p}\rangle_\infty = \sum_{k=0}^{p-1} q^{\gamma_k+pk}\langle -q^{\alpha_k+mp},-q^{\beta_k+np};q^y\rangle_\infty \tag{2.8c}$$

$$\langle -q^{3bm+2p},-q^{3bn+2p};q^{3p}\rangle_\infty = \sum_{k=0}^{p-1} q^{\gamma_k+pk}\langle -q^{\alpha_k+(m+n)p},-q^{\beta_k-(m-n)p};q^y\rangle_\infty. \tag{2.8d}$$

For all integers t, let

$$\sigma_t = \frac{t(t+1)y}{2} \qquad (y=3p^2),$$

so that equations (2.3) and (2.3′) state that

$$\langle -q^{x+ty};q^y\rangle_\infty = q^{(y-x)t-\sigma_t}\langle -q^x;q^y\rangle_\infty \qquad \text{and} \qquad \langle -q^{x-ty};q^y\rangle_\infty = q^{tx-\sigma_t}\langle -q^x;q^y\rangle_\infty.$$

Corollary 2. *The k-summands in equations (2.8a)–(2.8d) are all invariant under the transformation $k \to k+p$.*

Proof. We prove the statement for (2.8a), namely the general term

$$q^{\gamma_k}\langle -q^{\alpha_k}; q^y\rangle_\infty \langle -q^{\beta_k}; q^y\rangle_\infty.$$

Changing k to $k+p$, we directly find that

$$\alpha_{k+p} = \alpha_k + 3mp^2 = \alpha_k + my,$$
$$\beta_{k+p} = \beta_k + 3np^2 = \beta_k + ny,$$

so that

$$\begin{aligned}\langle -q^{\alpha_{k+p}}, -q^{\beta_{k+p}}; q^y\rangle_\infty &= \langle -q^{\alpha_k+my}, -q^{\beta_k+ny}; q^y\rangle_\infty \\ &= q^{\left[(y-\alpha_k)m-\sigma_m\right]+\left[(y-\beta_k)n-\sigma_n\right]}\langle -q^{\alpha_k}, -q^{\beta_k}; q^y\rangle_\infty\end{aligned}$$

Now observing that

$$\gamma_{k+p} = \gamma_k + \gamma_p + 3kp^2,$$

we simply compute that

$$\gamma_{k+p} + \left[(y-\alpha_k)m - \sigma_m\right] + \left[(y-\beta_k)n - \sigma_n\right] = \gamma_k + \tfrac{1}{2}p(3p+6k-1)(p-m^2-n^2) = \gamma_k,$$

and deduce that

$$q^{\gamma_{k+p}}\langle -q^{\alpha_{k+p}}, -q^{\beta_{k+p}}; q^y\rangle_\infty = q^{\gamma_k}\langle -q^{\alpha_k}, -q^{\beta_k}; q^y\rangle_\infty,$$

as claimed. The invariance of the terms in (2.8b)–(2.8d) are proved similarly. □

Definition 2. Throughout the remainder of the article, let μ satisfy

$$3m\mu \equiv 1 \pmod{p},$$

and define s via

$$3m\mu = 1 + ps.$$

2.1. **A key dissection lemma.** By the definitions of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$ and a_t it is clear that

$$\begin{aligned}\sum_{t=-\infty}^{\infty} a_t q^t = \langle -q^{3bm+p}, -q^{3bn+p}; q^{3p}\rangle_\infty - q^{bn}\langle -q^{3bm+p}, -q^{3bn+2p}; q^{3p}\rangle_\infty \\ - q^{bm}\langle -q^{3bm+2p}, -q^{3bn+p}; q^{3p}\rangle_\infty + q^{bm+bn}\langle -q^{3bm+2p}, -q^{3bn+2p}; q^{3p}\rangle_\infty,\end{aligned} \tag{2.9}$$

Lemma 4. *Fix r with $0 \le r \le p-1$, let κ satisfy*

$$3b\kappa m \equiv r \pmod{p}, \qquad 0 \le \kappa \le p-1,$$

and for this fixed κ let

$$\alpha = \alpha_\kappa, \quad \beta = \beta_\kappa, \quad \text{and} \quad \gamma = \gamma_\kappa.$$

Then the r-component of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$, namely $\sum_{t\in\mathbb{Z}} a_{tp+r}q^{tp+r}$, is equal to

$$\begin{aligned}&q^{\gamma}\langle -q^{\alpha}, -q^{\beta}; q^y\rangle_\infty - q^{\gamma+\delta}\langle -q^{\alpha-nsp^2}, -q^{\beta+msp^2}; q^y\rangle_\infty \\ &\quad - q^{\gamma+\xi}\langle -q^{\alpha-msp^2}, -q^{\beta-nsp^2}; q^y\rangle_\infty + q^{\gamma+\delta+\xi}\langle -q^{\alpha-msp^2-nsp^2}, -q^{\beta+msp^2-nsp^2}; q^y\rangle_\infty,\end{aligned} \tag{2.10}$$

where

$$\delta = -bnsp + \tfrac{1}{6}(1+ps)p^2 s,$$
$$\xi = -bmps - \kappa p^2 s + \tfrac{1}{6}(1+ps)p^2 s.$$

Proof. The equations of Corollary 1 yield that

$$\begin{aligned}\sum_{t=-\infty}^{\infty} a_t q^t &= \sum_k q^{\gamma_k}\langle -q^{\alpha_k}, -q^{\beta_k}; q^y\rangle_\infty - \sum_k q^{bn+\gamma_k}\langle -q^{\alpha_k+np}, -q^{\beta_k-mp}; q^y\rangle_\infty \\ &\quad - \sum_k q^{bm+\gamma_k+pk}\langle -q^{\alpha_k+mp}, -q^{\beta_k+np}; q^y\rangle_\infty \\ &\quad + \sum_k q^{b(m+n)+\gamma_k+pk}\langle -q^{\alpha_k+(m+n)p}, -q^{\beta_k+(m-n)p}; q^y\rangle_\infty \\ &=: \Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4,\end{aligned}$$

and within each Σ_i, the r-component is determined by some $0 \le k_i \le p-1$ that makes the "external q-exponent" equivalent to $r \pmod p$[1]. Specifically, since $\gamma_k \equiv 3bkm \pmod p$, for $1 \le i \le 4$ the r-component of Σ_i corresponds to $0 \le k_i \le p-1$ satisfying

(2.11a) $$3bk_1 m \equiv r \pmod p,$$

(2.11b) $$bn + 3bk_3 m \equiv r \pmod p,$$

(2.11c) $$bm + 3bk_2 m \equiv r \pmod p,$$

(2.11d) $$b(m+n) + 3bk_4 m \equiv r \pmod p,$$

respectively. Fixing κ such that

$$3b\kappa m \equiv r \pmod p$$

then, the quantities

(2.12) $$k_1 = \kappa, \quad k_2 = \kappa - n\mu, \quad k_3 = \kappa - m\mu, \quad \text{and} \quad k_4 = \kappa - m\mu - n\mu$$

satisfy (2.11a)–(2.11d), respectively, and with this fixed κ we let

$$\alpha = \alpha_\kappa, \quad \beta = \beta_\kappa, \quad \text{and} \quad \gamma = \gamma_\kappa.$$

It remains now to simply expand the k-th terms in each Σ_i using the respective k_i from (2.12). There is nothing to do when $k = k_1 = \kappa$, so let $k = k_2 = \kappa - n\mu$ and consider the k-summand of Σ_2. We find that

$$\begin{aligned}\alpha_{\kappa-n\mu} + np &= \alpha - nsp^2, \\ \beta_{\kappa-n\mu} - mp &= \beta + msp^2 - \mu y, \\ bn + \gamma_{\kappa-n\mu} &= \gamma + \delta_0\end{aligned}$$

where

$$\delta_0 := -bnsp + (1-3\kappa)\mu n + \tfrac{1}{2}\mu n(3\mu n - 1)p.$$

Thus, the r-component of Σ_2 is

$$q^{\gamma+\delta_0}\langle -q^{\alpha - nsp^2}, -q^{\beta+msp^2-\mu y}; q^y\rangle_\infty,$$

[1] Recall from (2.7a) and (2.7b) that $p \mid \alpha_k$ and $p \mid \beta_k$, respectively.

and we apply (2.3′) to simplify this to

$$q^{\gamma+\delta}\langle -q^{\alpha-nsp^2}, -q^{\beta+msp^2}; q^y\rangle_\infty, \qquad \text{with} \qquad \delta = -bnsp + \frac{p^2 s(1+ps)}{6}.$$

The analogous formulae for Σ_3 and Σ_4 similarly follow by direct computations and applications of (2.3′). □

3. The case $p \equiv 1 \pmod{12}$

The assumption that $p \equiv 1 \pmod{12}$, in particular the assumption that $p \equiv 1 \pmod 3$, implies that either m or n must be $0 \pmod 3$. As our arguments (previous and following) are symmetric in m and n, no generality is lost in assuming that $n \equiv 0 \pmod 3$. With this assumption, recalling that $3m\mu = 1 + ps$, in this section we set

$$N := \frac{ns}{3}.$$

The goal of this section is proof of the following theorem.

Theorem 3. *Let $p \equiv 1 \pmod{12}$ and $p = m^2 + n^2$, with m and n positive and such that $3|n$. Let b be a positive integer.*

(3) *If $m \equiv 1 \pmod 3$, then*

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) = \sum_{k=0}^{p-1} q^{\gamma_k} Q(q^{bp+kmp+\frac{(p-m-n)p}{6}}, q^{p^2})Q(q^{-knp+\frac{(p-m+n)p}{6}}, q^{p^2}). \tag{3.1}$$

(3) *If $m \equiv 2 \pmod 3$, then*

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) = \sum_{k=0}^{p-1} q^{\gamma_k} Q(q^{-bp-kmp+\frac{(p+m+n)p}{6}}, q^{p^2})Q(q^{knp+\frac{(p+m-n)p}{6}}, q^{p^2}). \tag{3.2}$$

Proof. In short, we aim to show that in the first case, all r-components ($0 \le r \le p-1$) of both sides of (3.1) are equal, and similarly for (3.2) in the second case. Thus, as in Lemma 4, fix r with $0 \le r \le p-1$, let κ satisfy

$$3b\kappa m \equiv r \pmod p, \qquad 0 \le \kappa \le p-1,$$

and for this κ let

$$\alpha = \alpha_\kappa, \quad \beta = \beta_\kappa, \quad \text{and} \quad \gamma = \gamma_\kappa.$$

By Lemma 4, the r-component of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$ is

$$\begin{aligned} &q^\gamma\langle -q^\alpha, -q^\beta; q^y\rangle_\infty - q^{\gamma+\delta}\langle -q^{\alpha-nsp^2}, -q^{\beta+msp^2}; q^y\rangle_\infty \\ &\quad - q^{\gamma+\xi}\langle -q^{\alpha-msp^2}, -q^{\beta-nsp^2}; q^y\rangle_\infty + q^{\gamma+\delta+\xi}\langle -q^{\alpha-msp^2-nsp^2}, -q^{\beta+msp^2-nsp^2}; q^y\rangle_\infty, \end{aligned} \tag{3.3}$$

where

$$\delta = -bnsp + \tfrac{1}{6}(1+ps)p^2 s \qquad \text{and} \qquad \xi = -bmps - \kappa p^2 s + \tfrac{1}{6}(1+ps)p^2 s.$$

Now under our assumptions that $3|n$ and $N = \frac{ns}{3}$, we may further reduce (3.3). In particular, as $nsp^2 = 3Np^2 = Ny$, (3.3) becomes

$$\begin{aligned}q^{\gamma}\langle -q^{\alpha}, -q^{\beta}; q^y\rangle_\infty &- q^{\gamma+\delta+[\alpha N-\sigma_N]}\langle -q^{\alpha}, -q^{\beta+msp^2}; q^y\rangle_\infty \\ &- q^{\gamma+\xi+[\beta N-\sigma_N]}\langle -q^{\alpha-msp^2}, -q^{\beta}; q^y\rangle_\infty \\ &+ q^{\gamma+\delta+\xi+\left[(\alpha-msp^2)N-\sigma_N\right]+\left[(\beta+msp^2)N-\sigma_N\right]}\langle -q^{\alpha-msp^2}, -q^{\beta+msp^2}; q^y\rangle_\infty,\end{aligned}$$

which we see can be factored as

$$\begin{aligned}q^{\gamma}&\left(\langle -q^{\alpha}; q^y\rangle_\infty - q^{\xi+[\beta N-\sigma_N]}\langle -q^{\alpha-msp^2}; q^y\rangle_\infty\right) \\ &\times\left(\langle -q^{\beta}; q^y\rangle_\infty - q^{\delta+[\alpha N-\sigma_N]}\langle -q^{\beta+msp^2}; q^y\rangle_\infty\right).\end{aligned} \tag{3.4}$$

We now pause to consider the right-hand sides of (3.1) and (3.2). Recalling that

$$Q(z,q) = \langle -z^3q; q^3\rangle_\infty - z\langle -z^3q^2; q^3\rangle_\infty,$$

if A and B are some quantities, then

$$\begin{aligned}Q(q^{A/3}, q^{p^2})Q(q^{B/3}, q^{p^2}) = &\left(\langle -q^{A+p^2}; q^y\rangle_\infty - q^{A/3}\langle -q^{A+2p^2}; q^y\rangle_\infty\right) \\ &\times\left(\langle -q^{B+p^2}; q^y\rangle_\infty - q^{B/3}\langle -q^{B+2p^2}; q^y\rangle_\infty\right).\end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5), we want to select A and B to i) equate the two expressions; and ii) to have [at least one of] $A/3$ and $B/3$ be integral. Evidently, to equate (3.4) and (3.5) we want

$$A = \alpha - p^2 \qquad \text{or} \qquad A = 2p^2 - \alpha,$$

and

$$B = \beta - p^2 \qquad \text{or} \qquad B = 2p^2 - \beta.$$

For our second condition, dividing these potential A and B by 3, we see that

$$\frac{\alpha - p^2}{3} = bp + \kappa mp + \frac{(p-m-n)p}{6} \qquad \text{and} \qquad \frac{2p^2-\alpha}{3} = -bp - \kappa mp + \frac{(p+m+n)p}{6},$$

and

$$\frac{\beta - p^2}{3} = \kappa np + \frac{(p+m-n)p}{6} \qquad \text{and} \qquad \frac{2p^2-\beta}{3} = -\kappa np + \frac{(p-m+n)p}{6}.$$

Thus, our choices for A and B are made to ensure at least one of the numerators in each pair is $0 \pmod 6$. Already we know that $p \pm m \pm n \equiv 0 \pmod 2$, so it remains only to consider $p \pm m \pm n \pmod 3$. We have assumed that $p \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, so our choices for A and B depend only on $m \pmod 3$.

Focusing on the case $m \equiv 1 \pmod 3$, we see at once that

$$\frac{\alpha - p^2}{3} = bp + \kappa mp + \frac{(p-m-n)p}{6} \qquad \text{and} \qquad \frac{2p^2-\beta}{3} = -\kappa n + \frac{(p-m+n)p}{6}$$

are integral. Then letting

$$A = \alpha - p^2 \qquad \text{and} \qquad B = 2p^2 - \beta$$

in (3.5), one may directly verify that (3.4) and (3.5) are equal, and we deduce that

$$\sum_{t=-\infty}^{\infty} a_{tp+r}q^{tp+r} = q^{\gamma_\kappa} Q\Big(q^{bp+\kappa mp+\frac{(p-m-n)p}{6}}, q^{p^2}\Big) Q\Big(q^{-\kappa np+\frac{(p-m+n)p}{6}}, q^{p^2}\Big).$$

As $0 \le r \le p-1$ was arbitrary, equation (3.1) follows at once, proving part 3 of the theorem.

For the case $m \equiv 2 \pmod 3$, we now have

$$\frac{2p^2-\alpha}{3} = -bp - kmp + \frac{(p+m+n)p}{6} \qquad \text{and} \qquad \frac{\beta - p^2}{3} = knp + \frac{(p+m-n)p}{6}$$

are integral, and letting

$$A = 2p^2 - \alpha \qquad \text{and} \qquad B = \beta - p^2$$

in (3.5), one may directly verify that (3.4) and (3.5) are equal. Thus, in this case the r-component of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$ is

$$\sum_{t=-\infty}^{\infty} a_{tp+r}q^{tp+r} = q^{\gamma_\kappa} Q\Big(q^{-bp-\kappa mp+\frac{(p+m+n)p}{6}}, q^{p^2}\Big) Q\Big(q^{\kappa np+\frac{(p+m-n)p}{6}}, q^{p^2}\Big),$$

and equation (3.2) similarly follows by summing over $0 \le \kappa \le p-1$, which completes the proof of the theorem. □

Corollary 3. *For $p \equiv 1 \pmod{12}$, and m, n and b as in Theorem 3, let*

$$w \equiv \bar{2}(m+n) \pmod p,$$

and let the sequence $\{a_n\}$ be as defined at (2.9). Then for all integers n one has

$$a_{pn+bw} = a_{pn+b(w-3b)} = 0.$$

Proof. Recalling from Lemma 2 that $\langle q^X; q^Y\rangle_\infty = 0$ whenever $Y > 0$ and $X \equiv 0 \pmod Y$, it is evident that $Q(q^X, q^Y) = 0$ under the same conditions; our aim is to apply this to the Q-factors in (3.1) and (3.2). First supposing that $m \equiv 1 \pmod 3$, by Theorem 3 the r-component of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$ is

$$\sum_{t=-\infty}^{\infty} a_{tp+r}q^{tp+r} = q^{3b\kappa m+\frac{\kappa(3\kappa-1)p}{2}} Q(q^{bp+\kappa mp+\frac{(p-m-n)p}{6}}, q^{p^2}) Q(q^{-\kappa np+\frac{(p-m+n)p}{6}}, q^{p^2}),$$

where $0 \le \kappa \le p-1$ satisfies $3b\kappa m \equiv r \pmod p$. We first show that all $a_{tp+bw} = 0$, so let $r \equiv bw \pmod p$ and let κ satisfy

(3.6) $$3b\kappa m \equiv bw \pmod p.$$

From relation (3.6) and the fact that $p = m^2 + n^2$, it is easy to check that

$$6\kappa n \equiv \bar{m}n(m+n) \equiv \bar{m}(mn - m^2) \equiv n - m \pmod p,$$

where $m\bar{m} \equiv 1 \pmod p$, so let

(3.7) $$6\kappa n = n - m + pK.$$

As such, we have

(3.8) $$Q(q^{-\kappa np+\frac{(p-m+n)p}{6}}, q^{p^2}) = Q(q^{\frac{(1-K)p^2}{6}}, q^{p^2}),$$

and so, to apply Lemma 2, it suffices to show that $K \equiv 1 \pmod 6$. From (3.7) and our assumption that $p \equiv 1 \pmod{12}$, evidently $K \equiv m - n \pmod 6$. Observing then that $m - n \equiv 1 \pmod 2$ (trivially) and that $m - n \equiv m \equiv 1 \pmod 3$, it follows at once that $K \equiv 1 \pmod 6$, whence (3.8) is identically 0. Thus, all $a_{jp+bw} = 0$ when $m \equiv 1 \pmod 3$, as claimed.

Turning to the quantities $a_{tp+b(w-3b)}$, now let $r \equiv 3b\kappa m \equiv b(w-3b) \pmod p$, so that $6\kappa m \equiv -6b + m + n \pmod p$, and write

$$6\kappa m = -6b + m + n + pK \tag{3.9}$$

for some K. Then

$$Q(q^{bp+\kappa mp+\frac{(p-m-n)p}{6}}, q^{p^2}) = Q(q^{\frac{(1+K)p^2}{6}}, q^{p^2}), \tag{3.10}$$

and now to apply Lemma 2 it suffices to have $K \equiv -1 \pmod 6$; reducing (3.9) (mod 6) we see that $K \equiv -(m+n) \pmod 6$, and our assumptions on m and n again lead us to the desired conclusion. Thus, expression (3.10) is equal to 0, and it follows that all $a_{tp+b(w-3b)} = 0$.

The arguments for the case $m \equiv 2 \pmod 3$ follow completely parallel arguments using (3.2) in place of (3.1), which completes the proof of the corollary. □

Example 1. Let $p = 13 = 2^2 + 3^3$ and $b = 5$ in Corollary 3, and let the sequence $\{a_t\}$ be defined by

$$Q(q^{10}, q^{13})Q(q^{15}, q^{13}) = \sum_{t=0}^{\infty} a_t q^t.$$

Then

$$w \equiv 9 \pmod{13}, \quad bw \equiv 6 \pmod{13}, \quad \text{and} \quad b(w-3b) \equiv 9 \pmod{13},$$

whence for all integers t, one has

$$a_{13t+6} = a_{13t+9} = 0.$$

3.1. **Sign patterns when $p \equiv 1 \pmod{12}$.** We recall that

$$Q(q^\ell, q^p) = (q^\ell, q^{p-\ell}; q^p)_\infty (q^{p+2\ell}, q^{p-2\ell}; q^{2p})_\infty (q^p; q^p)_\infty. \tag{3.11}$$

Lemma 5. *For a given integer ℓ, let*

$$\ell = r + hp \qquad \textit{with} \qquad 0 \le r \le p-1.$$

One has

$$\langle q^\ell; q^p\rangle_\infty = (-1)^h q^{-hr-\frac{h(h-1)p}{2}} \langle q^r; q^p\rangle_\infty, \tag{3.12}$$

$$Q(q^\ell, q^p) = q^{-3hr-\frac{h(3h-1)p}{2}} Q(q^r, q^p). \tag{3.13}$$

Proof. Supposing that $\ell > 0$, so that $h \ge 0$, one can easily check that

$$(q^{p-\ell}; q^p)_\infty = (-1)^h q^{-hr-\frac{h(h-1)p}{2}} (q^r; q^p)_h (q^{p-r}; q^p)_\infty, \tag{3.14a}$$

$$(q^{p-2\ell}; q^{2p})_\infty = (-1)^h q^{-2hr-h^2p} (q^{p+2r}; q^{2p})_h (q^{p-2r}; q^{2p})_\infty, \tag{3.14b}$$

where we have used the standard q-Pochhammer notation $(z;q)_h := \prod_{k=0}^{h-1}(1-zq^k)$. Equation (3.12) then follows from (3.14a) and the observation that

$$(q^r; q^p)_h (q^\ell; q^p)_\infty = (q^r; q^p)_\infty, \tag{3.14c}$$

and equation (3.13) follows from (3.14a)–(3.14c) and the observation that

$$(q^{p+2r};q^{2p})_h(q^{p+2\ell};q^{2p})_\infty = (q^{p+2r};q^{2p})_\infty.$$

When $\ell < 0$, it is convenient to instead write $\ell = r - hp$ with $h > 0$, and similar, direct computations again establish (3.12) and (3.13). □

Lemma 6. *For fixed integer ℓ, let*

$$\ell = r + hp \qquad \text{with} \qquad 0 \le r < p,$$

and define $\varepsilon = \varepsilon(\ell)$ via

$$\varepsilon = \begin{cases} 0 & 0 \le r < \frac{p}{2}, \\ 1 & \frac{p}{2} < r \le p-1. \end{cases}$$

Then for some set $S \subset \mathbb{Z}_{\ge 0}$, one has

$$Q(q^\ell, q^p) = (-1)^\varepsilon q^H \prod_{n\in S}(1-q^n), \tag{3.15}$$

where $H = \varepsilon(p-2r) - 3hr - \frac{h(3h-1)p}{2}$.

Proof. If $p \mid \ell$ then $Q(q^\ell, q^p) = 0 = (1-q^0)$, so we suppose that $(\ell, p) = 1$ and (consequently) that $r > 0$. By (3.11) and (3.13), we have

$$Q(q^\ell, q^p) = q^{-3hr - \frac{h(3h-1)p}{2}}(q^r, q^{p-r}, q^p; q^p)_\infty (q^{p+2r}; q^{2p})_\infty \times (q^{p-2r}; q^{2p})_\infty,$$

and the factor $(q^r, q^{p-r}, q^p; q^p)_\infty (q^{p+2r}; q^{2p})_\infty$ is a product of terms $(1-q^n)$ with n running over some sub*set* of $\mathbb{N}$ (thus, no factors are repeated), say

$$(q^r, q^{p-r}, q^p; q^p)_\infty (q^{p+2r}; q^{2p})_\infty = \prod_{n\in S_1}(1-q^n).$$

When $0 < r < \frac{p}{2}$, the remaining factor $(q^{p-2r}; q^{2p})_\infty$ is similarly of the form $\prod_{n\in S_2}(1-q^n)$ for some $S_2 \subset \mathbb{N}$. Noting that r, $-r$, $2r$, and $-2r$ must all be distinct (mod p) since $(r,p) = 1$, it is evident that S_1 and S_2 are mutually exclusive, and thus (3.15) follows in this case.

When $\frac{p}{2} < r < p$ we similarly have

$$(q^{p-2r}; q^{2p})_\infty = -q^{p-2r}(1-q^{2r-p})(q^{3p-2r}; q^{2p})_\infty = -q^{p-2r}\prod_{n\in S_2'}(1-q^n),$$

for some $S_2' \subset \mathbb{N}$. It is again easy to see that S_1 and S_2' are disjoint subsets of $\mathbb{N}$, and (3.15) again follows. □

Corollary 4. *Let $p \equiv 1 \pmod{12}$, $p = m^2 + n^2$ with $m, n > 0$ and $3|n$. Let b be a positive integer coprime to p and define the sequence $\{b_n\}$ by*

$$\frac{Q(q^{bm}, q^p)Q(q^{bn}, q^p)}{(q^p;q^p)_\infty^2} = \sum_{n=-\infty}^{\infty} b_n q^n. \tag{3.16}$$

Fix $0 \le r \le p-1$ and κ satisfying

$$3b\kappa m \equiv r \pmod{p}, \qquad 0 \le \kappa \le p-1.$$

For $i=1,2$ *and integers* ℓ_i, *let* r_i *be the "least positive residue of* $\ell_i \pmod p$*"; that is,* r_i *satisfies* $\ell_i \equiv r_i \pmod p$ *with* $0 \le r_i \le p-1$. *Finally, define* $\varepsilon_i = \varepsilon_i(\ell_i)$ *via*

$$\varepsilon_i = \begin{cases} 0 & 0 \le r_i < \frac{p}{2}, \\ 1 & \frac{p}{2} < r_i \le p-1. \end{cases}$$

(4) *If* $m \equiv 1 \pmod 3$, *let*

$$\ell_1 := b + \kappa m + \tfrac{1}{6}(p-m-n) \qquad \textit{and} \qquad \ell_2 := -\kappa n + \tfrac{1}{6}(p-m+n).$$

Then

$$(-1)^{\varepsilon_1+\varepsilon_2} b_{tp+r} \ge 0.$$

(4) *If* $m \equiv 2 \pmod 3$, *let*

$$\ell_1 := -b - \kappa m + \tfrac{1}{6}(p+m+n) \qquad \textit{and} \qquad \ell_2 := \kappa n + \tfrac{1}{6}(p+m-n).$$

Then

$$(-1)^{\varepsilon_1+\varepsilon_2} b_{tp+r} \ge 0.$$

Proof. Starting with the dissections in Theorem 3, we replace q by q^p in Lemma 6 and then apply said lemma to each term of the dissection. After dividing by $(q^p;q^p)_\infty^2$, we expand

$$(q^p;q^p)_\infty = (q^p, q^{2p}, \ldots, q^{p^2}, \ldots, q^{(2p-1)p}, q^{2p^2}; q^{2p^2})_\infty$$

and cancel all the q-products in the numerator of each resulting infinite product from Lemma 6. ☐

Example 2. For $p = 13 = 2^2 + 3^2$ and $b = 5$, the signs of the coefficients b_{13n+r}, $r = 0, 1, \ldots, 12$ are as shown in the following table (after ignoring any initial zeroes).

r	0	1	2	3	4	5	6	7	8	9	10	11	12
sign of b_{13t+r}	−1	1	1	1	−1	−1	0	1	1	0	1	1	−1

The zeros indicate the vanishing coefficients in the progressions $r \equiv 6, 9 \pmod{13}$, as shown in Example 1.

4. The case $p \equiv 5 \pmod{12}$

Lemma 7 (Winquist's identity [28])**.** *For complex* a, b, *and* q, *with* $|q| < 1$ *and* $a, b \neq 0$, *let*

$$W(a,b,q) = \langle a, b, ab, \tfrac{a}{b}; q\rangle_\infty (q;q)_\infty^{-2}. \tag{4.1}$$

Then

$$W(a,b,q) = \langle a^3, b^3 q; q^3\rangle_\infty - b\langle a^3, b^3q^2; q^3\rangle_\infty - \frac{a}{b}\langle a^3 q, b^3; q^3\rangle_\infty + \frac{a^2}{b}\langle a^3q^2, b^3; q^3\rangle_\infty. \tag{4.2}$$

We illustrate the "broad strokes" of our derivations. First, we recall that

$$\begin{aligned}\sum_{j=-\infty}^{\infty} a_{jp+r} q^{jp+r} &= q^\gamma \langle -q^\alpha, -q^\beta; q^y\rangle_\infty - q^{\gamma+\delta}\langle -q^{\alpha-nsp^2}, -q^{\beta+msp^2}; q^y\rangle_\infty \\ &\quad - q^{\gamma+\xi}\langle -q^{\alpha-msp^2}, -q^{\beta-nsp^2}; q^y\rangle_\infty + q^{\gamma+\delta+\xi}\langle -q^{\alpha-(m+n)sp^2}, -q^{\beta+msp^2-nsp^2}; q^y\rangle_\infty,\end{aligned} \tag{4.3}$$

where α, β, γ, δ, and ξ all depend on some fixed k satisfying $3bkm \equiv r \pmod p$. Next, if A and B are some quantities, then Winquist's identity (4.2) yields that

$$W(-q^{\frac{A}{3}}, -q^{\frac{B}{3}}, q^{p^2}) = \langle -q^A, -q^{B+p^2}; q^y\rangle_\infty + q^{\frac{B}{3}}\langle -q^A, -q^{B+2p^2}; q^y\rangle_\infty - q^{\frac{A-B}{3}}\langle -q^{A+p^2}, -q^B; q^y\rangle_\infty - q^{\frac{2A-B}{3}}\langle -q^{A+2p^2}, -q^B; q^y\rangle_\infty. \tag{4.4}$$

Our goal is to determine some A, B, and χ (all depending on r and k) such that

$$\sum_{j=-\infty}^{\infty} a_{jp+r} q^{jp+r} = q^\chi W(-q^{A/3}, -q^{B/3}, q^{p^2}), \tag{4.5}$$

and we approach this by changing the triple product terms $\langle -q^\bullet; q^y\rangle_\infty$ (using $\bullet$ to denote a generic quantity) in (4.3) to resemble the terms in (4.4), and then ensuring that "external" factors $q^\bullet$ match afterwards.

We now introduce a handful of assumptions and definitions. We recall that

$$3m\mu = 1 + ps,$$

and note that since $p \equiv 2 \pmod 3$, it follows that $s \equiv 1 \pmod 3$. Now suppose that

$$m \equiv -n \pmod 3,$$

and let us explicitly say that

$$m \equiv 2 \pmod 3 \quad \text{with} \quad ms = 2 + 3M \quad \text{and} \quad -ms = 1 - 3(M+1), \tag{4.6a}$$

$$n \equiv 1 \pmod 3 \quad \text{with} \quad ns = 1 + 3N \quad \text{and} \quad -ns = 2 - 3(N+1). \tag{4.6b}$$

It is convenient now to let

$$\tau := \frac{(m+n)s}{3}.$$

With this and relation (2.3′), the last term of (4.3) is

$$\langle -q^{\alpha-\tau y}, -q^{\beta-nsp^2+msp^2}; q^y\rangle_\infty = q^{\alpha\tau-\sigma_\tau}\langle -q^\alpha, -q^{\beta-nsp^2+msp^2}; q^y\rangle_\infty,$$

and we can rewrite the right-hand side of (4.3) as

$$q^\gamma\langle -q^\alpha, -q^\beta; q^y\rangle_\infty + q^{\gamma+\delta+\xi+[\alpha\tau-\sigma_\tau]}\langle -q^\alpha, -q^{\beta-nsp^2+msp^2}; q^y\rangle_\infty - q^{\gamma+\delta}\langle -q^{\alpha-nsp^2}, -q^{\beta+msp^2}; q^y\rangle_\infty - q^{\gamma+\xi}\langle -q^{\alpha-msp^2}, -q^{\beta-nsp^2}; q^y\rangle_\infty, \tag{4.7}$$

and this latter expression is now (somewhat) reminiscent of the right-hand side of (4.4). In particular, we are motivated to set

$$A := \alpha \qquad \text{and} \qquad B := \beta - nsp^2$$

in (4.7), which changes said expression to

$$q^\gamma\langle -q^A, -q^{B+nsp^2}; q^y\rangle_\infty + q^{\gamma+\delta+\xi+[\alpha\tau-\sigma_\tau]}\langle -q^A, -q^{B+msp^2}; q^y\rangle_\infty - q^{\gamma+\delta}\langle -q^{A-nsp^2}, -q^{B+\tau y}; q^y\rangle_\infty - q^{\gamma+\xi}\langle -q^{A-msp^2}, -q^B; q^y\rangle_\infty. \tag{4.8}$$

Now using (4.6a) and (4.6b), we have

$$q^\gamma\langle -q^A, -q^{B+nsp^2}; q^y\rangle_\infty = q^{\gamma+[(y-B-p^2)N-\sigma_N]}\langle -q^A, -q^{B+p^2}; q^y\rangle_\infty,$$

$$q^{\gamma+\delta+\xi+[\alpha\tau-\sigma_\tau]}\langle -q^A, -q^{B+msp^2}; q^y\rangle_\infty = q^{\gamma+\delta+\xi+[\alpha\tau-\sigma_\tau]+\left[(y-B-2p^2)M-\sigma_M\right]}\langle -q^A, -q^{B+2p^2}; q^y\rangle_\infty.$$

Comparing these with (4.4) and (4.5) suggests that we take

$$\chi_k = \gamma + [(y - B - p^2)N - \sigma_N], \tag{4.9}$$

which, we are able to simplify to the definition

$$\chi_k = 3bkm + p\omega_k + p(1-ns)(kn + \tfrac{(p+m-n(1+ps))}{6}). \tag{4.9$'$}$$

Continuing with (4.6a), (4.6b), and (4.9′), we find that (4.8) is

$$\begin{aligned} &= q^{\chi}\langle -q^{A}, -q^{B+p^2}; q^y\rangle_\infty + q^{\chi+\frac{B}{3}}\langle -q^{A}, -q^{B+2p^2}; q^y\rangle_\infty \\ &\quad - q^{\chi+\frac{A-B}{3}}\langle -q^{A+p^2}, -q^{B}; q^y\rangle_\infty - q^{\chi+\frac{2A-B}{3}}\langle -q^{A+2p^2}, -q^{B}; q^y\rangle_\infty, \\ &= q^{\chi} W(-q^{A/3}, -q^{B/3}, q^{p^2}) \end{aligned} \tag{4.10}$$

If, in place of (4.6a) and (4.6b), we instead assume that $m \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, our derivations are nearly identical. Indeed, although our initial "definition" of χ_k is different than (4.9), this different form simplifies to be identical to (4.9′), and we again reduce (4.8) to (4.10).

Summarizing the above derivations, we have the following lemma.

Lemma 8. *Let $p \equiv 5 \pmod{12}$ with $p = m^2 + n^2$, let $b > 0$, and suppose that*

$$m \equiv -n \pmod 3.$$

Fixing $0 \le r \le p-1$, let k satisfy $3bkm \equiv r \pmod p$. In addition, let

$$\chi_k := 3bkm + p\omega_k + p(1-ns)(kn + \tfrac{(p+m-n(1+ps))}{6}).$$

Then

$$\sum_{t=-\infty}^{\infty} a_{tp+r} q^{tp+r} = q^{\chi_k} W(-q^{\alpha_k/3}, -q^{(\beta_k - nsp^2)/3}, q^{p^2}).$$

4.1. **The case $m \equiv n \pmod 3$.** Suppose that $m \equiv n \pmod 3$, let

$$\tau := \frac{(m-n)s}{3},$$

and suppose that

$$m \equiv 1 \pmod 3 \quad \text{and} \quad ms = 1 + 3M, \quad -ms = 2 - 3(M+1), \tag{4.11a}$$

$$n \equiv 1 \pmod 3 \quad \text{and} \quad ns = 1 + 3N, \quad -ns = 2 - 3(N+1). \tag{4.11b}$$

Let

$$A := \alpha_k - msp^2 \qquad \text{and} \qquad B := \beta_k,$$

so that (4.3) becomes

$$\begin{aligned} &q^{\gamma}\langle -q^{A+msp^2}, -q^{B}; q^y\rangle_\infty + q^{\gamma+\delta+\xi}\langle -q^{A-nsp^2}, -q^{B+\tau y}; q^y\rangle_\infty \\ &\quad - q^{\gamma+\delta}\langle -q^{A+\tau y}, -q^{B+msp^2}; q^y\rangle_\infty - q^{\gamma+\xi}\langle -q^{A}, -q^{B-nsp^2}; q^y\rangle_\infty. \end{aligned} \tag{4.12}$$

This time, we find that

$$q^{\gamma}\langle -q^{A+msp^2}, -q^{B}; q^y\rangle_\infty = q^{\gamma+\left[(y-A-p^2)M - \sigma_M\right]}\langle -q^{A+p^2}, -q^{B}; q^y\rangle_\infty,$$

$$q^{\gamma+\delta+\xi}\langle -q^{A-nsp^2}, -q^{B+\tau y}; q^y\rangle_\infty = q^{\gamma+\delta+\xi+\left[(A+2p^2)(N+1)-\sigma_{N+1}\right]+\left[(y-B)\tau-\sigma_\tau\right]}\langle -q^{A+2p^2}, -q^{B}; q^y\rangle_\infty,$$

and we are motivated to define (cf. (4.9))

$$\chi_k^* := \gamma + (y - A - p^2)M - \sigma_M.$$

After expanding and simplifying, we find (cf. (4.9′)) that

$$\chi_k^* = 3bkm + p\omega_k + p(1-ms)\left(b + km + \tfrac{p-m(1+ps)-n}{6}\right),$$

and ultimately that (4.12) is

$$\begin{aligned}
&= q^{\chi_k^*}\langle -q^{A+p^2}, -q^B; q^y\rangle_\infty + q^{\chi_k^*+\frac{A}{3}}\langle -q^{A+2p^2}, -q^B; q^y\rangle_\infty \\
&\quad - q^{\chi_k^*+\frac{B-A}{3}}\langle -q^A, -q^{B+p^2}\rangle_\infty - q^{\chi_k^*+\frac{2B-A}{3}}\langle -q^A, -q^{B+2p^2}\rangle_\infty \\
&= q^{\chi_k^*}W(-q^{B/3}, -q^{A/3}, q^{p^2}),
\end{aligned}$$

with $A = \alpha_k - msp^2$ and $B = \beta_k$.

As in the case $m \equiv -n \pmod 3$, assuming that $m \equiv n \equiv 2 \pmod 3$ in place of (4.6a) and (4.6b) ultimately yields that (4.12) is equal to $q^{\chi_k^*}W(-q^{B/3}, -q^{A/3}, q^{p^2})$, and thus we have established the following analogue to Lemma 8.

Lemma 9. *Let $p \equiv 5 \pmod{12}$ with $p = m^2 + n^2$, let $b > 0$, and suppose that*

$$m \equiv n \pmod 3.$$

Fixing $0 \le r \le p-1$, let k satisfy $3bkm \equiv r \pmod p$, and let

$$\chi_k^* := 3bkm + p\omega_k + p(1-ms)\left(b + km + \tfrac{p-m(1+ps)-n}{6}\right).$$

Then

$$\sum_{t=-\infty}^{\infty} a_{tp+r}q^{tp+r} = q^{\chi_k^*}W(-q^{\beta_k/3}, -q^{(\alpha_k - msp^2)/3}, q^{p^2}).$$

Theorem 4. *Let $p \equiv 5 \pmod{12}$ with $p = m^2 + n^2$, and let $b > 0$.*

(4) *If $m \equiv -n \pmod 3$, then defining*

$$\chi_k := 3bkm + p\omega_k + p(1-ns)(kn + \tfrac{(p+m-n(1+ps))}{6}),$$

one has

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) = \sum_{k=0}^{p-1} q^{\chi_k}W(-q^{\alpha_k/3}, -q^{(\beta_k - nsp^2)/3}, q^{p^2}). \tag{4.13}$$

(4) *If $m \equiv n \pmod 3$, then defining*

$$\chi_k^* := 3bkm + p\omega_k + p(1-ms)\left(b + km + \tfrac{p-m(1+ps)-n}{6}\right),$$

one has

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) = \sum_{k=0}^{p-1} q^{\chi_k^*}W(-q^{\beta_k/3}, -q^{(\alpha_k - msp^2)/3}, q^{p^2}). \tag{4.14}$$

As a preliminary to proving the next corollary, we note that

$$\langle q^x; q^{p^2}\rangle_\infty = 0 \quad \text{whenever} \quad x \equiv 0 \pmod{p^2}$$

Corollary 5. *Let p, m, n and b be as in Theorem 4, and let $m\bar{m} \equiv n\bar{n} \equiv 1 \pmod p$. Let the sequence $\{a_t\}$ be defined by*

$$Q(q^{bm}, q^p)Q(q^{bn}, q^p) = \sum_{t=0}^{\infty} a_t q^t, \tag{4.15}$$

and let

$$w \equiv \bar{2}(m+n) \pmod{p}.$$

Then for all integers t, one has

$$a_{tp+bw(1-3b\bar{m})} = a_{tp+bw(1-3b\bar{n})} = 0.$$

Proof. First suppose that $m \equiv -n \pmod 3$. Fixing $0 \le r \le p-1$, the r-component of $Q(q^{bm}, q^{bn})$ is

$$\begin{aligned} &q^{\chi_k} W(-q^{\alpha_k/3}, -q^{(\beta_k - nsp^2)/3}, q^{p^2}) \\ &\quad = \frac{q^{\chi_k}\langle -q^{\alpha_k/3}, -q^{(\beta_k-nsp^2)/3}; q^{p^2}\rangle_\infty}{(q^{p^2}; q^{p^2})_\infty^2} \times \langle q^{(\alpha_k+\beta_k-nsp^2)/3}, q^{(\alpha_k-\beta_k+nsp^2)/3}; q^{p^2}\rangle_\infty, \end{aligned} \tag{4.16}$$

where $0 \le k \le p-1$ satisfies

$$3bkm \equiv r \pmod p,$$

and, by Lemma 2, this component (4.16) will be identically zero if either

$$\alpha_k + \beta_k \equiv 0 \pmod{p^2} \qquad \text{or} \qquad \alpha_k - \beta_k \equiv 0 \pmod{p^2}.$$

Expanding these congruences using the definitions of α_k and β_k, we quickly find that said congruences are equivalent to the relations

$$3b - n + 3k(m+n) \equiv 0 \pmod p \qquad \text{and} \qquad 3b - m + 3k(m-n) \equiv 0 \pmod p. \tag{4.17}$$

Because $p = m^2 + n^2$, it is not possible that $m \pm n \equiv 0 \pmod p$, so we may solve these congruences for $k \pmod p$. In particular, using the facts that $m\bar{n} \equiv -\bar{m}n \pmod p$, that

$$\overline{m+n} \equiv \bar{2}(\bar{m}+\bar{n}) \pmod p, \qquad \text{and that} \qquad \overline{m-n} \equiv \bar{2}(\bar{m}-\bar{n}) \pmod p,$$

we rearrange (4.17) to find that

$$3k \equiv \bar{2}n(\bar{m}+\bar{n})(1-3b\bar{n}) \pmod p \qquad \text{and} \qquad 3k \equiv \bar{2}m(\bar{m}-\bar{n})(1-3b\bar{m}) \pmod p,$$

respectively, which are equivalent to

$$3bkm \equiv \bar{2}b(m+n)(1-3b\bar{n}) \pmod p \quad \text{and} \quad 3bkm \equiv \bar{2}b(m+n)(1-3b\bar{m}) \pmod p,$$

respectively. Thus, one has $a_{tp+r} = 0$ for all t when

$$r \equiv bw(1-3b\bar{n}) \pmod p \qquad \text{or} \qquad r \equiv bw(1-3b\bar{m}) \pmod p,$$

as claimed. The proof is nearly identical when $m \equiv n \pmod 3$, after making the necessary changes. □

Example 3. Let $p = 17 = 4^2 + 1^2$ and $b = 2$ in Corollary 5. Then

$$w \equiv 11 \pmod{17}, \quad bw(1-3b\bar{m}) \equiv 6 \pmod{17}, \quad \text{and} \quad bw(1-3b\bar{n}) \equiv 9 \pmod{17},$$

whence for all integers t, one has

$$a_{17t+6} = a_{17t+9} = 0.$$

Corollary 6. *Maintaining the assumptions of Corollary 5, again let*

$$w \equiv \bar{2}(m+n) \pmod{p}.$$

Then for all integers t, the quantities a_{tp+bw} and $a_{tp+b(w-3b)}$ are even; that is, one has

$$a_{pt+bw} \equiv a_{pt+b(w-3b)} \equiv 0 \pmod{2}. \tag{4.18}$$

Proof. The proof is similar to that of the previous corollary. Supposing that $m \equiv -n \pmod{3}$, for fixed $0 \le r \le p-1$ the r-component of (4.13) has factors

$$\langle -q^{\alpha_k/3}; q^{p^2}\rangle_\infty \qquad \text{and} \qquad \langle -q^{(\beta_k - nsp^2)/3}; q^{p^2}\rangle_\infty \tag{4.19}$$

coming from $W(-q^{\alpha_k/3}, -q^{(\beta_k-nsp^2)/3}, q^{p^2})$, where k satisfies $3bkm \equiv r \pmod{p}$.

If $\alpha_k \equiv 0 \pmod{p^2}$ or $\beta_k - nsp^2 \equiv 0 \pmod{p^2}$, then the corresponding triple product in (4.19) has a factor $1-(-q^0) = 2$. It is easily seen that these congruences are equivalent to

$$6b + 6km - (m+n) \equiv 0 \pmod{p} \qquad \text{and} \qquad 6kn + m - n \equiv 0 \pmod{p},$$

which hold when

$$3bkm \equiv b(w-3b) \pmod{p} \qquad \text{and} \qquad 3bkm \equiv w \pmod{p},$$

respectively. Thus, (4.18) thus holds in the case $m \equiv -n \pmod{3}$; the proof of (4.18) when $m \equiv n \pmod{3}$ follows a nearly identical argument, mutatis mutandis. ☐

Example 4. Let $p = 17 = 4^2 + 1^2$ and $b = 2$ in Corollary 6. Then

$$w \equiv 11 \pmod{17}, \quad bw \equiv 5 \pmod{17}, \quad \text{and} \quad b(w-3b) \equiv 10 \pmod{17},$$

whence

$$a_{17t+5} \equiv a_{17t+10} \equiv 0 \pmod{2} \qquad (t \ge 0).$$

Indeed, this is reflected in the computations

$$\{a_{17t+5} : t \ge 0\} = \{0, 0, 0, 0, -2, -4, -8, -16, -28, -48, -82, -132, -210, -328, -502, \dots\},$$
$$\{a_{17t+10} : t \ge 0\} = \{2, 4, 10, 20, 40, 72, 130, 220, 368, 594, 948, 1474, 2270, 3428, 5128, \dots\}.$$

4.2. **Sign patterns when $p \equiv 5 \pmod{12}$.**

Corollary 7. *Let p, m, n, b and s as in Theorem 4. Define the sequence $\{b_n\}$ by*

$$\frac{Q(q^{bm}, q^p)Q(q^{bn}, q^p)}{(q^p; q^p)_\infty^2} = \sum_{n=-\infty}^{\infty} b_n q^n. \tag{4.20}$$

Fix an $0 \le r \le p-1$ and let $0 \le k \le p-1$ be such that $3bkm \equiv r \pmod{p}$.

(7) *If $m \equiv -n \pmod{3}$, define*

$$\ell = b + k(m+n) + p - \tfrac{1}{3}n(1+ps),$$
$$\ell' = b + k(m-n) - \tfrac{1}{3}(m+n) + \tfrac{1}{3}n(1+ps).$$

Then

$$(-1)^{[\ell/p]+[\ell'/p]} b_{tp+r} \ge 0.$$

(7) $m \equiv n \pmod 3$, *define*

$$\ell = b + k(m+n) + p + \tfrac{1}{3}(m-n) - \tfrac{1}{3}m(1+ps),$$
$$\ell' = -b - k(m-n) + \tfrac{1}{3}m(1+ps).$$

Then

$$(-1)^{[\ell/p]+[\ell'/p]} b_{tp+r} \geq 0.$$

Proof. We recall from equation (3.12) that if $\ell = r + hp$ with $0 \leq r \leq p-1$, then

$$\langle q^\ell; q^p\rangle_\infty = (-1)^h q^{-hr-\frac{h(h-1)p}{2}} \langle q^r; q^p\rangle_\infty, \tag{4.21}$$

and certainly

$$\langle q^r; q^p\rangle_\infty = \prod_{n\in S}(1-q^n) \qquad \text{for some } S \subset \mathbb{Z}_{\geq 0}. \tag{4.22}$$

Supposing that $m \equiv -n \pmod 3$, by Theorem 4 the r-component of $Q(q^{bm}, q^p)Q(q^{bn}, q^p)$ is

$$\begin{aligned} &q^{\chi_k} W(-q^{\alpha_k/3}, -q^{(\beta_k - nsp^2)/3}, q^{p^2}) \\ &= \frac{q^{\chi_k}\langle -q^{\alpha_k/3}, -q^{(\beta_k-nsp^2)/3}; q^{p^2}\rangle_\infty}{(q^{p^2}; q^{p^2})^2_\infty} \times \langle q^{(\alpha_k+\beta_k-nsp^2)/3}, q^{(\alpha_k-\beta_k+nsp^2)/3}; q^{p^2}\rangle_\infty. \end{aligned} \tag{4.23}$$

Letting

$$\ell = \frac{\alpha_k + \beta_k - nsp^2}{3p} = b + k(m+n) + p - \frac{n(1+ps)}{3},$$
$$\ell' = \frac{\alpha_k - \beta_k + nsp^2}{3p} = b + k(m-n) - \frac{m+n}{3} + \frac{n(1+ps)}{3},$$

we have

$$\langle q^{(\alpha_k+\beta_k-nsp^2)/3}, q^{(\alpha_k-\beta_k+nsp^2)/3}; q^{p^2}\rangle_\infty = \langle q^{p\ell}, q^{p\tilde{\ell}}; q^{p^2}\rangle_\infty,$$

and, letting $h := [\ell/p]$ and $h' := [\ell'/p]$, by (4.21) and (4.22) we have

$$\langle q^{p\ell}, q^{p\ell'}; q^{p^2}\rangle_\infty = (-1)^{h+h'} q^{H+H'} \prod_{n\in S}(1-q^{pn}) \prod_{n\in S'}(1-q^{pn}), \tag{4.24}$$

where S and S' are subsets of $\mathbb{Z}_{\geq 0}$, and

$$H = -hr - \frac{h(h-1)p}{2} \qquad \text{and} \qquad H' = -h'r - \frac{h'(h'-1)p}{2}.$$

Now dividing (4.23) by $(q^p; q^p)^2_\infty$, the products over S and S' in (4.24) are cancelled by $(q^p; q^p)^2_\infty$, and part 7 of the result follows, since the remaining factors in (4.23) have series expansions with all coefficients nonnegative. The assertion of part 7 is proved similarly. ☐

Example 5. For $p = 17 = 4^2 + 1^2$ and $b = 2$, the signs of the coefficients b_{17n+r}, $r = 0, 1, \ldots, 16$ are (ignoring initial zeroes) as shown in the table.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
sign of b_{17t+r}	1	−1	−1	1	1	−1	0	−1	−1	0	1	1	−1	−1	1	−1	1

The zeroes indicate the vanishing coefficients in the progressions $r \equiv 6, 9 \pmod{17}$, as shown in Example 3.

5. Combinatorial interpretations

Here we give two examples of how to interpret the dissection results combinatorially. For any set $S \subset \mathbb{N}$ and any $n > 0$, let $D_S(n)$ denote the number of *even*-length, distinct-part partitions using only parts from S, minus the number of *odd*-length, distinct-part partitions using parts from S; in addition, set $D_S(0) = 1$.

Example 6. If we set $p = 13$, $n = 3$, $m = 2$ and $b = 1$ in Theorem 3, then after removing any negative exponents we get

$$
\begin{aligned}
(5.1)\quad Q(q^2,q^{13})Q(q^3,q^{13}) &= Q(q^{26},q^{169})^2 - q^{27}Q(q^{52},q^{169})Q(q^{78},q^{169}) - q^2Q(q^{39},q^{169})^2 \\
&- q^3Q(q^{13},q^{169})Q(q^{39},q^{169}) + q^{17}Q(q^{52},q^{169})Q(q^{65},q^{169}) + q^5Q(q^{26},q^{169})Q(q^{52},q^{169}) \\
&+ q^{19}Q(1,q^{169})Q(q^{65},q^{169}) - q^7Q(q^{13},q^{169})Q(q^{52},q^{169}) - q^{34}Q(q^{65},q^{169})Q(q^{78},q^{169}) \\
&+ q^9Q(1,q^{169})Q(q^{13},q^{169}) - q^{23}Q(q^{39},q^{169})Q(q^{78},q^{169}) \\
&- q^{24}Q(q^{13},q^{169})Q(q^{78},q^{169}) + q^{12}Q(q^{26},q^{169})Q(q^{65},q^{169}).
\end{aligned}
$$

As has already been shown, the $r = 6$ and $r = 9$ components of the above are zero, due in both cases to their factors of $Q(1, q^{169})$. Here we give a combinatorial interpretation of these identically zero components, and we relate the nonzero components of the dissection to the corresponding pieces in the series on the left of (5.1) in a combinatorial way.

If both sides of (5.1) are divided by $(q^{13}; q^{13})_\infty^2$, then the left side becomes

$$
(5.2)\qquad \frac{Q(q^2,q^{13})Q(q^3,q^{13})}{(q^{13};q^{13})_\infty^2} = (q^2,q^3,q^{10},q^{11};q^{13})_\infty(q^7,q^9,q^{17},q^{19};q^{26})_\infty = \sum_{n=0}^{\infty} D_A(n)q^n,
$$

where

$$
A = \{m \in \mathbb{N} : \ m \equiv \pm 2, \pm 3, \pm 7, \pm 9, \pm 10, \pm 11 \pmod{26}\}.
$$

Thus we have that $D_A(13n + 6) = D_A(13n + 9) = 0$ for all $n \geq 0$. As an example, for $n = 74 = 9 + 5(13)$ there are 158 even-length, distinct-part partitions of 74 using parts from A, and 158 odd-length, distinct-part partitions of 74 using parts from A.

On the other hand, if we consider the part of the dissection containing powers of q with exponents congruent to 5 (mod 13), then

$$
\sum_{n=0}^{\infty} D_A(13n+5)q^{13n+5} = \frac{q^5Q(q^{26},q^{169})Q(q^{52},q^{169})}{(q^{13};q^{13})_\infty^2},
$$

whereby

$$
\begin{aligned}
&\sum_{n=0}^{\infty} D_A(13n+5)q^n \\
&\quad = \frac{(q^2,q^{11},q^{13};q^{13})_\infty(q^9,q^{17};q^{26})_\infty}{(q;q)_\infty} \times \frac{(q^4,q^9,q^{13};q^{13})_\infty(q^5,q^{21};q^{26})_\infty}{(q;q)_\infty} \\
&\quad = \sum_{n=0}^{\infty} p_{13,5}(n),
\end{aligned}
$$

where

$$
p_{13,5}(n) := \#\{(\pi_1,\pi_2) : \ \pi_1 \in \mathcal{B}, \pi_2 \in \mathcal{C}, |\pi_1| + |\pi_2| = n\},
$$

where $\mathcal{B}$ is the set of partitions with parts in B, $\mathcal{C}$ is the set of partitions with parts in C, and

$$\begin{aligned} B &= \{m \in \mathbb{N} : \ m \not\equiv 0, \pm 2, \pm 9, \pm 11, 13 \pmod{26}, \\ C &= \{m \in \mathbb{N} : \ m \not\equiv 0, \pm 4, \pm 5, \pm 9, 13 \pmod{26}\}, \end{aligned}$$

and $|\pi_i|$ denotes the sum of the parts in the partition π_i (or equivalently, $|\pi_i|$ is the integer that π_i partitions). Note that $\pi_i = \{\,\}$, $i = 1$ or $i = 2$, is allowed, in which case $|\pi_i| = 0$.

For example, consider $n = 96 = 13(7) + 5$. There are 609 partitions of 96 into an even number of distinct parts from A, 547 partitions of 96 into an odd number of distinct parts from A, so that $D_A(96) = 609 - 547 = 62$, and there are also 62 bi-partitions/partition pairs (π_1, π_2), with $\pi_2 \in \mathcal{B}$, $\pi_2 \in \mathcal{C}$, and $|\pi_1| + |\pi_2| = 7$.

Example 7. Setting $p = 17 = 4^2 + 1^1$ and $b = 2$ in Theorem 4, then, after removing any negative exponents, we get

$$\begin{aligned} (5.3)\quad Q(q^2, q^{17})Q(q^8, q^{17}) &= \frac{1}{(q^{289}; q^{289})_\infty^2}\Big[W(-q^{136}, -q^{68}, q^{289}) - qW(-q^{119}, -q^{51}, q^{289}) \\ &- q^2W(-q^{119}, -q^{34}, q^{289}) + q^3W(-q^{119}, -q^{68}, q^{289}) + q^{21}W(-q^{85}, -q^{17}, q^{289}) \\ &- q^{73}W(-q^{34}, -1, q^{289}) + q^{23}W(-q^{85}, -q^{85}, q^{289}) - q^{24}W(-q^{136}, -q^{119}, q^{289}) \\ &- q^8W(-q^{102}, -q^{34}, q^{289}) - q^{43}W(-q^{51}, -q^{51}, q^{289}) + q^{10}W(-q^{136}, -1, q^{289}) \\ &+ q^{28}W(-q^{68}, -q^{51}, q^{289}) - q^{12}W(-q^{102}, -q^{17}, q^{289}) - q^{13}W(-q^{136}, -q^{102}, q^{289}) \\ &+ q^{14}W(-q^{102}, -q^{85}, q^{289}) - q^{66}W(-q^{34}, -q^{17}, q^{289}) + q^{33}W(-q^{68}, -q^{17}, q^{289})\Big]. \end{aligned}$$

It was shown in Corollary 3 that the coefficients in the arithmetic progressions $6, 9 \pmod{17}$ are zero, and this can be also be seen from (5.3), since $W(x, x, q^r) = 0$ for any x and any positive integer r. Here the combinatorial interpretation of the coefficients that vanish in arithmetic progressions is similar to that given in Example 6, but the combinatorial interpretation that arises from equating the non-zero components of the dissection to the corresponding pieces in the series on the left of (5.3) is different.

If both sides of (5.3) are divided by $(q^{17}; q^{17})_\infty^2$, then the left side becomes

$$(5.4)\qquad \frac{Q(q^2, q^{17})Q(q^8, q^{17})}{(q^{17}; q^{17})_\infty^2} = (q^2, q^8, q^9, q^{15}; q^{17})_\infty (q, q^{13}, q^{21}, q^{33}; q^{34})_\infty = \sum_{n=0}^{\infty} D_A(n)q^n,$$

where

$$A = \{m \in \mathbb{N} : \ m \equiv \pm 1, \pm 2, \pm 8, \pm 9, \pm 13, \pm 15 \pmod{34}\}.$$

Thus we have that $D_A(17n + 6) = D_A(17n + 9) = 0$ for all $n \geq 0$. For example for $n = 77 = 17(4) + 9$, there are 56 partitions of 77 into an even number of distinct parts from A, and 56 partitions of 77 into an odd number of distinct parts from A.

On the other hand if we consider the part of the dissection containing powers of q with exponent $\equiv 2 \pmod{17}$, then

$$\sum_{n=0}^{\infty} D_A(17n + 2)q^{17n+2} = \frac{-q^2W(-q^{119}, -q^{34}, q^{289})}{(q^{289}; q^{289})_\infty^2 (q^{17}; q^{17})_\infty^2},$$

whereby

$$\sum_{n=0}^{\infty} D_A(17n+2)q^n$$
$$= -(-q^2, -q^7, -q^{10}, -q^{15}; q^{17})_\infty \frac{(q^5, q^{12}, q^{17}; q^{17})_\infty}{(q;q)_\infty} \times \frac{(q^8, q^9, q^{17}; q^{17})_\infty}{(q;q)_\infty}$$
$$= -\sum_{n=0}^{\infty} p_{17,2}(n),$$

where

$$p_{17,2}(n) := \#\{(\pi_1, \pi_2, \pi_3) : \pi_2 \in \mathcal{B}, \pi_2 \in \mathcal{C}, \pi_3 \in \mathcal{D}, |\pi_1| + |\pi_2| + |\pi_3| = n\},$$

where $\mathcal{B}$ is the set of partitions with distinct parts in B, $\mathcal{C}$ is the set of partitions with parts in C, $\mathcal{D}$ is the set of partitions with parts in D, and

$$B = \{m \in \mathbb{N} : m \equiv \pm 2, \pm 7 \pmod{17}\},$$
$$C = \{m \in \mathbb{N} : m \not\equiv 0, \pm 5 \pmod{17}\},$$
$$D = \{m \in \mathbb{N} : m \not\equiv 0, \pm 8 \pmod{17}\}.$$

Note that, as in the previous example, $\pi_i = \{\}$ is allowed, and again in such a case we have $|\pi_i| = 0$.

For example, consider $n = 189 = 17(11)+2$. There are 5013 partitions of 189 into an even number of distinct parts from A, 5989 partitions of 189 into an odd number of distinct parts from A, so that $D_A(189) = 5013 - 5989 = -976$, and there are also 976 partition triples (π_1, π_2, π_3), with $\pi_1 \in \mathcal{B}$, $\pi_2 \in \mathcal{C}$, $\pi_3 \in \mathcal{D}$ and $|\pi_1| + |\pi_2| + |\pi_3| = 11$.

6. Concluding remarks

The present paper was motivated by the experimental discovery of products of the form $Q(q^r, q^p)Q(q^s, Q^p)$ with coefficients that vanished in arithmetic progressions, where r and s are positive integers and $p \equiv 1 \pmod 4$ a prime. After further experimentation, the connection between r and s was discovered, namely that

$$(r, s) = (mb, nb), \text{ where } p = m^2 + n^2, \text{ some } b \in \mathbb{N}.$$

However, this is not the end of the story, as experiment suggests the existence of products of the form $Q(q^r, q^p)Q(q^s, Q^p)Q(q^t, q^p)$ with coefficients that also vanish in arithmetic progressions. For example, experiment suggests that if

$$Q(q, q^{13})Q(q^3, q^{13})Q(q^4, q^{13}) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$a_{13n+2} = a_{13n+4} = a_{13n+10} = 0, \ \forall n \geq 0.$$

Furthermore, it would appear that in the case of products of three quintuple products, $Q(q^i, q^p)Q(q^j, q^p)Q(q^k, q^p)$, this vanishing coefficient phenomenon is not restricted to primes of the form $p \equiv 1 \pmod 4$. Experiment also suggests that if

$$Q(q^2, q^{19})Q(q^3, q^{19})Q(q^5, q^{19}) = \sum_{n=0}^{\infty} b_n q^n,$$

then

$$b_{19n+4} = b_{19n+5} = b_{19n+16} = 0, \ \forall n \geq 0.$$

We leave it to the reader to spot a possibly required condition, suggested by the two examples, on the i, j and k in $Q(q^i, q^p)Q(q^j, q^p)Q(q^k, q^p)$ for vanishing to occur.

References

[1] Alladi, K.; Gordon B. *Vanishing coefficients in the expansion of products of Rogers-Ramanujan type.* Proc. Rademacher Centenary Conference, (G. E. Andrews and D. Bressoud, Eds.), Contemp. Math. **166**, (1994), 129–139.

[2] Andrews, G. E.; Bressoud, D. M. *Vanishing coefficients in infinite product expansions.* J. Austral. Math. Soc. Ser. A **27** (1979), no. 2, 199—202.

[3] N.D. Baruah and M. Kaur, *Some results on vanishing coefficients in infinite product expansions.* Ramanujan J. **53** (2020), no. 3, 551–568.

[4] Bringmann, K.; Han, G.; Heim, B.; Kane, B. *Vanishing properties of Fourier coefficients of holomorphic η-quotients*, preprint

[5] Channabasavayya; Dasappa, R. *On a generalization of vanishing coefficients in two q-series expansions.* J. Ramanujan Math. Soc. **39** (2024), no. 2, 187–192.

[6] Chern, S.; Tang, D. *Vanishing coefficients in quotients of theta functions of modulus five.* Bull. Aust. Math. Soc. **102** (2020), no. 3, 387–398.

[7] Chern, S.; Tang, D. *General coefficient-vanishing results associated with theta series.* Adv. in Appl. Math. *159* (2024), Paper No. 102742, 66 pp.

[8] Cooper, S. *The quintuple product identity.* Int. J. Number Theory **2** (2006), no. 1, 115–161.

[9] Daniels, T. *Vanishing coefficients in two q-series related to Legendre-signed partitions.* Res. Number Theory **10** (2024), no. 4, Paper No. 81, 12 pp.

[10] Daniels, T.; Mc Laughlin, J. *Vanishing coefficients in the series expansion of some infinite products involving Legendre symbols.* - submitted.

[11] Hirschhorn, M. D. *The power of q: A personal journey*, Springer, 2017.

[12] Huber, T.; McLaughlin, J.; Ye, D. *Dissection of the quintuple product, with applications*, J. Combinatorial Theory, Series A, **219** (2026), no. 106122.

[13] Kaur, M.; Vanda. *Results on vanishing coefficients in infinite q-series expansions for certain arithmetic progressions mod* 7. Ramanujan J. **58** (2022), no. 1, 269–289.

[14] Liu, J.-C. *On the vanishing coefficients of odd powers of Ramanujan's theta functions.* Ramanujan J. **65** (2024), no. 1, 45–2.

[15] Liu, Z.-G.; Yang, X.-M. On the Schröter formula for theta functions. *Int. J. Number Theory* **5** (2009), no. 8, 1477–1488.

[16] McLaughlin, J. *Further results on vanishing coefficients in infinite product expansions*, J. Aust. Math. Soc. **98** (2015), no. 1, 69–77.

[17] McLaughlin, J. *New infinite q-product expansions with vanishing coefficients.* Ramanujan J **55**, 733–760 (2021).

[18] McLaughlin, J. and Zimmer, P. *Further results on Vanishing Coefficients in infinite products of the form* $(q^b, q^{p-b}; q^p)^3_\infty (q^{jb}, q^{2p-jb}; q^{2p})_\infty$ Int. J. Number Theory **18** (2022), no. 8, 1863–1885.

[19] Ono, K.; Robins, S. Superlacunary cusp forms. *Proc. Amer. Math. Soc.* **123** (1995), no. 4, 1021–1029.

[20] Richmond, B.; Szekeres, G. *The Taylor coefficients of certain infinite products.* Acta Sci. Math. (Szeged) **40** (1978), no. 3–4, 347—369.

[21] Somashekara, D. D.; Thulasi, M. B. *Results on vanishing coefficients in certain infinite q-series expansions.* Ramanujan J. **60** (2023), no. 2, 355–369.

[22] Tang, D. *Vanishing coefficients in some q-series expansions.* Int. J. Number Theory **15** (2019), no. 4, 763–773.

[23] Tang, D. *Vanishing coefficients on four quotients of infinite product expansions.* Bull. Aust. Math. Soc. **100** (2019), no. 2, 216–224.

[24] Tang, D. *Vanishing coefficients in three families of products of theta functions.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2023), no. 1, Paper No. 36, 11 pp.

[25] Tang, D. *Vanishing coefficients in three families of products of theta functions. II.* Results Math. **78** (2023), no. 4, Paper No. 124, 28 pp.

[26] Tang, D. *Vanishing coefficients in powers of theta functions with odd moduli.* Rocky Mountain J. Math. **54** (2024), no. 1, 261–267.

[27] Vanda; Kaur, M. *Vanishing coefficients of q^{5n+r} and q^{11n+r} in certain infinite q-product expansions.* Ann. Comb. **26** (2022), no. 3, 533–557.

[28] Winquist, L. *An elementary proof of* $p(11n+6) \equiv 0 (\bmod 1)1)$, J. Combin. Theory **6** (1969), 56-–59.

[29] Ernest X. W. Xia and Alice X. H. Zhao. *Generalizations of Hirschhorn's Results on Two Remarkable q-Series Expansions*, Experimental Mathematics, 31(3):878–882, 2022. https://doi.org/10.1080/10586458.2020.1712565.

Purdue University, 150 N University St, W Lafayette, IN 47907
Email address: daniel84@purdue.edu

School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, Texas 78539, USA
Email address: timothy.huber@utrgv.edu

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383
Email address: jmclaughlin2@wcupa.edu

Department of Mathematical Sciences, and School of AI and Liberal Arts, Beijing, Normal-Hong Kong Baptist University, Zhuhai 519082, Guangdong, People's Republic of China
Email address: dongxiye@bnbu.edu.cn