Identically Vanishing Coefficients in the Series Expansion of Lacunary Eta Quotients

Seminar in Partition Theory, q-Series and Related Topics,

.

James Mc Laughlin
(Joint work with Tim Huber and Dongxi Ye)

West Chester University, PA

Web page of James Mc Laughlin

jmclaughlin2@wcupa.edu

March 7 2024

Overview

Background and Notation

q-products

$$
\begin{aligned}
(a;q)_\infty &= (1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots, \\
(a_1,a_2,\ldots a_n;q)_\infty &= (a_1;q)_\infty(a_2;q)_\infty\ldots(a_n;q)_\infty.
\end{aligned}
$$

For $|q|<1$,
$$
\begin{aligned}
(q;q)_\infty &:= (1-q)(1-q^2)(1-q^3)\cdots \\
f_1 &:= (q;q)_\infty \\
f_j &:= (q^j;q^j)_\infty
\end{aligned}
$$

The series $\sum_{n=0}^{\infty} c(n)q^n$ is *lacunary* if

$$
\lim_{x\to\infty} \frac{|\{n \mid 0\leq n\leq x, c(n)=0\}|}{x} = 1.
$$

q-products Continued

Serre: for <u>even</u> positive integers s, f_1^s is lacunary if and only if

$$s \in \{2, 4, 6, 8, 10, 14, 26\}.$$

For <u>odd</u> positive integers s f_1^s lacunar,y for $s = 1$ and $s = 3$, but nothing that is conclusive is known otherwise.

Definition: An *eta quotient* is a finite product of the form $\prod_j f_j^{n_j}$, for some integers $j \in \mathbb{N}$ and $n_j \in \mathbb{Z}$.

One could also ask about more general eta quotients that are lacunary.

Origins of the Present Work

A Result of Han and Ono

Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$f_1^8 =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_3^3}{f_1} =: \sum_{n=0}^{\infty} b(n)q^n. \tag{1}$$

Theorem

(*Han and Ono, 2011*) *Assuming the notation above, we have that*

$$a(n) = 0 \Longleftrightarrow b(n) = 0. \tag{2}$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $3n + 1$ has a prime factor p of the form $p = 6k + 5$ for some integer k, with odd exponent.

The Result of Han and Ono in More Detail

$$
\begin{aligned}
f_1^8 = 1 - 8q + 20q^2 - 70q^4 + 64q^5 + 56q^6 - 125q^8 - 160q^9 + 308q^{10} \\
+ 110q^{12} - 520q^{14} + 57q^{16} + 560q^{17} + 182q^{20} + \dots, \\
\frac{f_3^3}{f_1} = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + 2q^9 + 2q^{10} + 2q^{12} \\
+ 2q^{14} + 3q^{16} + 2q^{17} + 2q^{20} + \dots .
\end{aligned}
$$

Notice that the two series vanish for the same powers of q, namely q^n with $n = 3, 7, 11, 13, 15, 18, 19 \dots$.

Further, for any n in this list, $3n + 1$ has a prime factor p of the form $p = 6k + 5$ with odd exponent.

(For example, for $n = 11$, $3n + 1 = 3(11) + 1 = 34 = 2(17^1)$ and $17 = 6(2) + 5$.)

Series with identically vanishing coefficients

The eta quotient f_1^8 was shown by Serre to be lacunary.

Do similar situations exist for the other powers of f_1 that are lacunary?

We first introduce some additional notation.

If $A(q)$ and $B(q)$ are two functions for which the coefficients in the series expansions satisfy the condition (2) in the theorem

$$a(n) = 0 \iff b(n) = 0,$$

then for ease of discussion, we say that *the coefficients vanish identically*, or that $A(q)$ and $B(q)$ have *identically vanishing coefficients*.

Series with identically vanishing coefficients II

Theorem 1 motivated the speaker to investigate experimentally if similar results held for other pairs of eta quotients.

This was done using some simple *Mathematica* programs.

What was discovered as a result of these computer algebra experiments is summarized as follows.

Other eta quotients with identically vanishing coefficients I

Let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \quad (3)$$

For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n) q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n) q^n. \quad (4)$$

Then, for each pair, $a(n) = 0 \iff b(n) = 0$, with criteria for when exactly this happens (Serre's criteria).

Other eta quotients with identically vanishing coefficients II

For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1} \right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6} \right) \right\} \tag{5}$$

$a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$.

The proofs needed the theory of modular forms (enter Tim Huber and later Dongxi Ye).

Later: The results above on identically vanishing coefficients appear to be just "the tip of the iceberg".

Brief Comment on the method of proof

Brief outline of method of proof:

- Apply a dilation $q \to q^m$ and multiply by q^j (some integers m and j) to turn the second eta quotient into a modular form.

- Use the LMFDB to express the resulting modular form as a linear combination of CM forms (by a result of Serre on lacunary forms, and also using the Sturm bound to verify the equality).

- Use a result of Ribet to express the CM forms as linear combinations of theta series of the form $\sum_{m,n}(m+n\sqrt{-D})^k q^{m^2+Dn^2}$, where D is a positive integer and the m and n run over all the integers or certain arithmetic progressions (allows the coefficient b_p of q^p to be computed explicitly in terms of the m and n in $p = m^2 + Dn^2$).

- Use the multiplicativity of the coefficients in the CM forms, and the recursive formula for prime powers to determine information about a general coefficient b_n (and in particular, when $b_n = 0$).

Some Sample Proofs

Some Sample Proofs

A Proof involving Triangular numbers I

Theorem

Define the sequences $\{a(n)\}$ *and* $\{b(n)\}$ *as follows:*

$$f_1^6 =: \sum_{n=0}^{\infty} a(n) q^n, \qquad \frac{f_2^4}{f_1^2} =: \sum_{n=0}^{\infty} b(n) q^n.$$

Then

$$a(n) = 0 \iff b(n) = 0. \tag{6}$$

Moreover, we have that $a(n) = b(n) = 0$ *precisely for those non-negative* n *for which* $ord_p(4n+1)$ *is odd for some prime* $p \equiv 3 \pmod 4$*.*

The proof of this theorem does not involve CM forms and theta series (so different from most other proofs in the paper).
Serre: $a(n) = 0$ if and only if $4n+1$ has a prime factor $p \equiv 3 \pmod 4$ with odd exponent, so it suffices to show $b(n) = 0$ under the same conditions.

A Proof involving Triangular numbers II

Fact:

$$\frac{f_2^2}{f_1}=\sum_{n=0}^{\infty}q^{n(n+1)/2}, \Longrightarrow \frac{f_2^4}{f_1^2}=\sum_{m,n=0}^{\infty}q^{m(m+1)/2+n(n+1)/2}=\sum_{k=0}^{\infty}b(k)q^k.$$

Let

$$t(n)=\frac{n(n+1)}{2}, \quad n=0,1,2,3,\dots,$$

denote the n-th triangular number. Let

$$T_2=\{t(m)+t(n)|m,n\geq 0\},$$

the set of non-negative integers representable as a sum of two triangular numbers. Thus $b(k)=0$ if and only if $k\notin T_2$.

A Proof involving Triangular numbers III

There is the following criterion of Ewell (1992):

Proposition

A positive integer n can be written as a sum of two triangular numbers if and only if when $4n+1$ is expressed as a product of prime-powers, every prime factor $p \equiv 3 \pmod 4$ occurs with even exponent.

Thus $b(n) \neq 0$ if and only if when $4n+1$ is expressed as a product of prime-powers, every prime factor $p \equiv 3 \pmod 4$ occurs with even exponent.
Alternatively, $b(n) = 0$ if and only if when $4n+1$ is expressed as a product of prime-powers, some prime factor $p \equiv 3 \pmod 4$ occurs with odd exponent.

However, this is exactly Serre's criterion for $a(n) = 0$.

An Example of the More Usual Kind of Proof I

Theorem

Define the sequences $\{a(n)\}$ *and* $\{b(n)\}$ *by*

$$f_1^4 =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_1^8}{f_2^2} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then $a(n) = 0 \iff b(n) = 0$.
Moreover, we have that $a(n) = b(n) = 0$ *precisely for those non-negative* n *for which* $ord_p(6n+1)$ *is odd for some prime* $p \equiv 5 \pmod 6$.

Serre: $a(n) = 0$ precisely for those non-negative n for which $ord_p(6n+1)$ is odd for some prime $p \equiv 5 \pmod 6$, so it is sufficient to show $b(n) = 0$ under the same conditions.

An Example of the More Usual Kind of Proof II

Next, apply a dilation $q \to q^6$ to each eta quotient, and then multiply by q:

$$q\, f_6^4 =: \sum_{n=0}^{\infty} a(n) q^{6n+1}, \qquad q\, \frac{f_6^8}{f_{12}^2} = \sum_{n=0}^{\infty} b(n) q^{6n+1} =: \sum_{n=0}^{\infty} b_n^* q^n.$$

The form $q\, f_6^8/f_{12}^2$ is a lacunary form of weight 3 and level 144, and hence by a criterion of Serre is a linear combination of CM forms of the same weight and level.

The next step is to head to the LMFDB (The L-functions and modular forms database (LMFDB)) to look for these CM forms.

Method: Use *Mathematica* to set up a linear combination of the series for qf_6^8/f_{12}^2 and those for all the CM forms from the LMFDB of weight 3 and level 144, and look for a non-trivial linear combination that equals 0.

An Example of the More Usual Kind of Proof III

If we write $q = e^{2\pi i z}$, with z in the upper half of the complex plane,

$$q\,\frac{f_6^8}{f_{12}^2} = \frac{\eta^8(6z)}{\eta^2(12z)} = q - 8q^7 + 22q^{13} - 16q^{19} - 25q^{25} + 24q^{31} + 26q^{37} + 48q^{43} - 143q^{49} + 74q^{61} + 32q^{67} + 46q^{73} - 40q^{79} - 176q^{91} - 2q^{97} + \dots$$

Next, let $S(q)$ denote the CM form of weight 3 and level 144 labelled 144.3.g.c in the LMFDB.
Then $S(q)$ has q-series expansion

$$\begin{aligned} S(q) = q - 8i\sqrt{3}q^7 + 22q^{13} - 16i\sqrt{3}q^{19} - 25q^{25} + 24i\sqrt{3}q^{31} \\ + 26q^{37} + 48i\sqrt{3}q^{43} - 143q^{49} + 74q^{61} + 32i\sqrt{3}q^{67} \\ + 46q^{73} - 40i\sqrt{3}q^{79} - 176i\sqrt{3}q^{91} - 2q^{97} + \dots . \end{aligned} \tag{7}$$

An Example of the More Usual Kind of Proof IV

Let $\bar{S}(q)$ denote the conjugate form $(i \to -i)$. By comparing coefficients up to the Sturm bound, one gets that

$$\frac{\eta^8(6z)}{\eta^2(12z)} = \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{-3}}\right)S(q) + \left(1-\frac{1}{\sqrt{-3}}\right)\bar{S}(q)\right].$$

Let the sequences $\{s_n\}$ and $\{\bar{s}_n\}$ be defined by

$$S(q) = \sum_{n=0}^{\infty} s_n q^n, \qquad \bar{S}(q) = \sum_{n=0}^{\infty} \bar{s}_n q^n. \tag{8}$$

Observe that

$$b^*_{12n+1} = s_{12n+1} = \bar{s}_{12n+1}, \qquad b^*_{12n+7} = \frac{s_{12n+7}}{i\sqrt{3}} = -\frac{\bar{s}_{12n+7}}{i\sqrt{3}}. \tag{9}$$

Note that $s_2 = s_3 = 0$, and if p is a prime, $p \equiv 2 \pmod 3$ (or $p \equiv 5 \pmod 6$), then $s_p = 0$.

An Example of the More Usual Kind of Proof V

The recurrence formula for s_n at prime powers is

$$s_{p^k} = s_p s_{p^{k-1}} - \chi(p) p^2 s_{p^{k-2}}, \tag{10}$$

where $\chi(p) = (-1)^{(p-1)/2}$.

This gives that if $p \equiv 5 \pmod 6$ is prime, so that $s_p = 0$, then

$$|s_{p^{2k}}| = p^{2k} \neq 0 \text{ and } s_{p^{2k+1}} = 0,$$

for all integers $k \geq 0$.

An Example of the More Usual Kind of Proof VI

The multiplicative property, $s_{uv} = s_u s_v$ if $\gcd(u, v) = 1$, gives that if

$$6n + 1 = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r},$$

then

$$s_{6n+1} = s_{p_1^{n_1}} s_{p_2^{n_2}} \dots s_{p_r^{n_r}},$$

and hence if some $p_i \equiv 5 \pmod 6$ and the corresponding n_i is odd, then $s_{6n+1} = 0$ and hence $b_n = 0$ (so giving half the proof).

The remainder of the proof is to show that if the factorization of $6n + 1$ is otherwise, then $s_{6n+1} \neq 0$, and hence $b_n \neq 0$.

What is necessary is to show $s_{p^n} \neq 0$ if $p \equiv 1 \pmod 6$ and $n \geq 1$.

Warning: Personal "black box" (Ribet's results on representing CM forms by theta series) ahead.

An Example of the More Usual Kind of Proof VII

For any Dirichlet character ϕ of conductor m, a newform $f(z)$ is said to have CM by ϕ if $a(p)\phi(p) = a(p)$ for all $p \nmid Nm$.

Such an $f(z)$ is also called a CM newform by ϕ.

It is known that a CM newform of weight $k \geq 2$ exists only if ϕ is a quadratic character associated to some quadratic field K.

In such case, $f(z)$ is also called a CM newform by K.

Ribet gives a full characterization of such newforms and justifies that any CM newform of weight $k \geq 2$ by a quadratic field K must come from a Hecke character ψ_K associated to K and be of the form

$$f(z) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ \textit{integral}}} \psi_K(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{\frac{k-1}{2}} q^{\mathcal{N}(\mathfrak{a})},$$

where $\mathcal{N}(\cdot)$ denotes the norm of an ideal.

An Example of the More Usual Kind of Proof VIII

In particular, when K is imaginary of discriminant $-d < 0$ and class number 1, one has that $f(z)$ must be a linear combination of the generalized theta series

$$\sum_{\alpha \in \beta + \mathfrak{m}} \alpha^{k-1} q^{\mathcal{N}(\alpha)} \quad \text{over } \beta \in (\mathcal{O}_K/\mathfrak{m})^\times$$

for some integral ideal $\mathfrak{m}$ with $\mathcal{N}(\mathfrak{m}) = N/d$.

An Example of the More Usual Kind of Proof IX

Next, following on from the material on the previous slides, define the theta series

$$H_1 = \sum_{m,n}(-6n+1+(4m-2n)\sqrt{-3})^2 q^{((-6n+1)^2+3(4m-2n)^2)}, \tag{11}$$

$$H_2 = \sum_{m,n}(-6n+5+(4m-2n)\sqrt{-3})^2 q^{((-6n+5)^2+3(4m-2n)^2)},$$

$$H_3 = \sum_{m,n}(-6n-2+(4m-2n+3)\sqrt{-3})^2 q^{((-6n-2)^2+3(4m-2n+3)^2)},$$

$$H_4 = \sum_{m,n}(-6n+2+(4m-2n+3)\sqrt{-3})^2 q^{((-6n+2)^2+3(4m-2n+3)^2)}.$$

One has that

$$S(q) = H_1 - H_2 - H_3 + H_4, \qquad \bar{S}(q) = H_1 - H_2 + H_3 - H_4. \tag{12}$$

An Example of the More Usual Kind of Proof X

These theta series will be used to get explicit expressions for s_p, when $p \equiv 1 \pmod 6$. For what purpose?
Recall

$$s_{p^k} = s_p s_{p^{k-1}} - \chi(p) p^2 s_{p^{k-2}},$$

From this one has, for any positive integer k, that

$$s_{p^k} \equiv s_p s_{p^{k-1}} \equiv \cdots \equiv (s_p)^k \pmod p.$$

Thus, if it can be shown that $s_p \not\equiv 0 \pmod p$, then $s_{p^k} \not\equiv 0 \pmod p$, and hence $s_{p^k} \neq 0$.

This would complete the proof that $s_{6n+1} = 0 \iff 6n+1$ has a prime factor $p \equiv 5 \pmod 6$ with odd exponent.

This in turn gives that $b(n) = 0 \iff 6n+1$ has a prime factor $p \equiv 5 \pmod 6$ with odd exponent.

An Example of the More Usual Kind of Proof XI

The quadratic forms in the exponents of each of the theta series represent a prime $p \equiv 1 \pmod 6$ in at most two ways (and in some cases not at all).

Hence (after some tedious calculations) it can be shown that if $p \equiv 1 \pmod{12}$, $p = x^2 + 3y^2$, then

$$s_p = \pm 2(x^2 - 3y^2) \equiv \pm 4x^2 \not\equiv 0 \pmod p.$$

Likewise, if $p \equiv 7 \pmod{12}$, $p = x^2 + 3y^2$, then

$$s_p = \pm 4xy\sqrt{-3} \not\equiv 0 \pmod p.$$

By the remarks on the previous slide, this completes the proof.

Let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \right.$$
$$\left. \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \tag{13}$$

For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n)q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n)q^n. \tag{14}$$

Then, for each pair, $a(n) = 0 \iff b(n) = 0$, with criteria for when exactly this happens (Serre's criteria).

Recap II

For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1} \right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6} \right) \right\} \tag{15}$$

$a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$.

How Extensive is this Phenomenon?

Notice that each of the triples

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}, \frac{f_2^8}{f_1^2}\right), \left(f_1^{26}, \frac{f_3^9}{f_1}, \frac{f_2^{16}}{f_1^6}\right) \right\} \quad (16)$$

have identically vanishing coefficients.

Q. How extensive is this phenomenon of eta quotients with identically vanishing coefficients?

A. Quite extensive.

The Results of More Extensive Investigations

The Results of More Extensive Investigations

Motivated by what we discovered (described in the previous section) we extended the search for the phenomenon described.

For ease of notation, for a function $E(q)=\sum_{n\geq 0} e_n q^n$ we write

$$E_{(0)} := \{n \in \mathbb{N} : e_n = 0\}$$

It was found that if $A(q)$ is any one of f_1^r, $r = 4, 6, 8, 10, 14$ and 26 or $f_1^3 f_2^3$ (the simplest case of an infinite family of lacunary eta quotients stated by Ono and Robins), then in each case there were a large numbers of eta quotients $B(q)$ such that $A_{(0)} = B_{(0)}$.

Further, in each case there were also many other eta quotients $C(q)$ such that $A_{(0)} \subsetneqq C_{(0)}$.

We describe what was found in some detail in the case of f_1^4.

The Case of f_1^4 I

Our limited search in the case of f_1^4 found a total of 72 eta quotients $B(q)$ for which it appeared ${f_1^4}_{(0)} = B_{(0)}$.

In addition, this search found 78 additional eta quotients with the property that for each such eta quotient $C(q)$, it seemed ${f_1^4}_{(0)} \subsetneq C_{(0)}$.

Moreover, it appears that all 150 eta quotients $B(q)$ may be organized into 19 collections (labelled I - XIX in what follows) in a tree-like structure by partially ordering the corresponding $B_{(0)}$ by inclusion.

The Case of f_1^4 II

Table 1: Eta quotients with vanishing behaviour similar to f_1^4

Collection	# of eta quotients	Collection	# of eta quotients
I	72	II *	4
III †	2	IV	6
V †	2	VI *	4
VII *	6	VIII *	8
IX *	4	X	4
XI	14	XII †	2
XIII †	2	XIV †	2
XV	4	XVI †	2
XVII	4	XVIII †	2
XIX *	6		

The Case of f_1^4 III

Thus, for example, all 14 eta quotients in the collection labelled XI, where

$$XI = \left\{ \frac{f_2 f_8^{14} f_{12}^2}{f_4^6 f_6 f_{16}^5 f_{24}}, \frac{f_6 f_8^{13}}{f_2 f_4^3 f_{12} f_{16}^5}, \frac{f_2^2 f_8 f_{12}^2}{f_4^2 f_{24}}, \frac{f_8^{11}}{f_2^2 f_{16}^5}, \frac{f_4^4 f_{12}^2}{f_2^2 f_8 f_{24}}, \frac{f_2^2 f_8^{13}}{f_4^6 f_{16}^5}, \frac{f_4^{15} f_6 f_{24}}{f_2^5 f_8^5 f_{12}^3}, \right.$$
$$\left. \frac{f_2^5}{f_6}, \frac{f_2^2 f_4^4}{f_8^2}, \frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}, \frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}, \frac{f_4^{10}}{f_2^2 f_8^4}, \frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}, \frac{f_4^4 f_6^7}{f_2^3 f_{12}^4} \right\}$$

appeared to have identically vanishing coefficients.

Collection I is the collection containing f_1^4.

* - has been proven that all eta quotients in the corresponding group have identically vanishing coefficients.

† - group members trivially have identically vanishing coefficients.

The relationships between eta quotients in different collections is illustrated in Figure 1.

The Case of f_1^4 IV

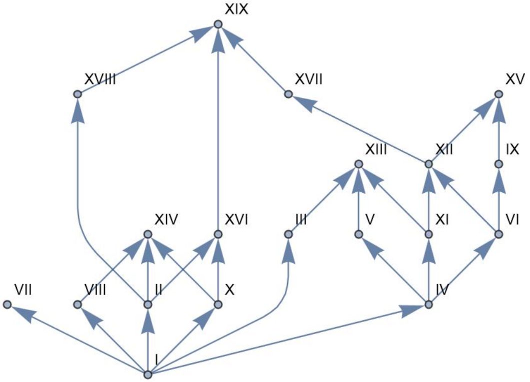

Figure: The grouping of the 150 eta-quotients in Table 1, which have vanishing coefficient behaviour similar to f_1^4

General Inclusion Results

General Inclusion Results I

Recall the amount of work necessary to show that if $A(q) = f_1^4$ and $B(q) = f_1^8/f_2^2$, then

$$A_{(0)} = B_{(0)}.$$

Clearly this method is not practical to prove the many hundreds of cases of identically vanishing coefficients that are suggested by experiment. Even if someone did decide to attempt this, the LMFDB (The L-functions and modular forms database (LMFDB)) is incomplete, and many of the CM forms needed to express a particular eta quotient are likely to be absent.

In the paper that describes this deeper investigation (the paper that has the various tables and figures shown/mentioned earlier) we do give some proofs, mostly to illustrate the various methods that may be used.

However, we were able to prove some quite general inclusion results. To describe those, we consider the figure for the collection related to f_1^6:

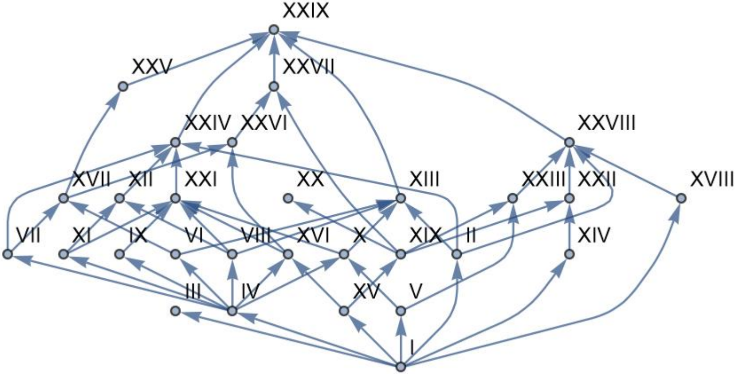

Figure: The grouping of the 172 eta-quotients which have vanishing coefficient behaviour similar to f_1^6

General Inclusion Results III

Recall that f_1^6 is in collection I, so that the figure suggests that if $A(q) = f_1^6$, and $B(q)$ is any one of the 172 eta quotients in the various collections, then

$$A_{(0)} \subseteq B_{(0)}.$$

We were in fact able to prove the above statement.

Similarly, we were able to prove the corresponding statements for the collections involving, respectively, f_1^4, f_1^8, f_1^{10}, f_1^{14}, f_1^{26} and $f_1^3 f_2^3$.

In each case, two general approaches gave us most of the results, and a small number of sporadic cases had to be treated separately.

General Inclusion Results IV

We illustrate one of the methods by an example for the $A(q) := f_1^6$ table. Recall:

Lemma

The equation $x^2 + y^2 = n$, $n > 0$ has integral solutions if and only if $\operatorname{ord}_p n$ is even for every prime $p \equiv 3 \pmod 4$. When that is the case, the number of solutions is

$$\prod_{p \equiv 1 \pmod 4} (1 + \operatorname{ord}_p n).$$

Serre's criterion: If

$$f_1^6 = \sum_{n=0}^{\infty} a_n q^n, \tag{17}$$

one has that $a_n = 0$ if and only if $4n + 1$ has a prime factor $p \equiv 3 \pmod 4$ with odd exponent.

There are various eta quotients which have expressions as single-sum theta series. For our present purposes,

$$\frac{f_1^2}{f_2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \qquad q\frac{f_{48}^{13}}{f_{24}^5 f_{96}^5} = \sum_{m=1}^{\infty} \left(\frac{-6}{m}\right) m q^{m^2}.$$

Consider the following eta quotient in collection XXI

$$B(q) := \frac{f_4^2 f_{12}^{13}}{f_6^5 f_8 f_{24}^5} =: \sum_{n=0}^{\infty} b_n q^n.$$

After applying the dilation $q \to q^4$ and multiplying by q:

$$\sum_{n=0}^{\infty} b_n q^{4n+1} = \frac{f_{16}^2}{f_{32}} \times q\frac{f_{48}^{13}}{f_{24}^5 f_{96}^5} = \sum_{\substack{m=1 \\ n=-\infty}}^{\infty} m(-1)^n \left(\frac{-6}{m}\right) q^{m^2+16n^2}.$$

We can now show $A_{(0)} \subseteq B_{(0)}$ (equivalently, $a_n = 0 \Longrightarrow b_n = 0$).

General Inclusion Results VI

Suppose $a_N = 0$, for some integer N.

Then, by Serre's criterion, $4N+1$ has a prime factor $p \equiv 3 \pmod 4$ with odd exponent.

By the lemma, $4N+1$ is not representable as a sum of two squares, and in particular not by $m^2+16n^2 = m^2+(4n)^2$.

Thus the coefficient of q^{4N+1} in

$$\sum_{\substack{m=1\\ n=-\infty}}^{\infty} m(-1)^n \left(\frac{-6}{m}\right) q^{m^2+16n^2}$$

is zero.

Hence $b_N = 0$, and thus $A_{(0)} \subseteq B_{(0)}$.

Remark: All the work in finding representations of eta quotients in the tables as products of two eta quotients with theta series expansions was performed by *Mathematica*.

General Inclusion Results VII

The other general result involved expressing eta quotients of weight ≥ 2 involved expressing the appropriate dilations of the eta quotients as certain sums over ideals in various number fields (recall earlier when expressing the CM forms as linear combinations of theta series).

General Inclusion Results VIII

The 5 exceptional cases (let any one of them be denoted by $B(q)$) in the 172 eta quotients in the f_1^6 table were treated as follows. Define

$$h_1(q;j,k)=\sum_{m,n=0}^{\infty} q^{(24m+j)^2+(24n+k)^2},$$

$$h_2(q;j,k)=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} q^{(24m+j)^2+4(24n+k)^2},$$

$$g_1(q;j,k)=\sum_{m,n=0}^{\infty} q^{(20m+j)^2+(20n+k)^2},$$

$$g_2(q;j,k)=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} q^{(20m+j)^2+4(20n+k)^2}.$$

Then $qB(q^4)$ is a linear combination of $h_i(q;j,k)$ for $i\in\{1,2\}$ and $0\le j,k\le 23$ and $g_i(q;,j,k)$ for $i\in\{1,2\}$ and $0\le j,k\le 19$. Since each exponent is a sum of two squares, the same argument can be used.

Dissection Methods

Interlude: The "$q \to -q$" Partner of an Eta Quotient

We often make the substitution $q \to -q$ in an eta quotient but wish to write the resulting product also as an eta quotient.

This leads to the following frequently employed identity:

$$f_1 = (q;q)_\infty \overset{q \to -q}{\longrightarrow} (-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty} = \frac{f_2^3}{f_1 f_4} \tag{18}$$

If $g(q) = f(-q)$, for simplicity we will call $g(q)$ the "$q \to -q$ partner" of $f(q)$.

The relevance in the present context is that a function and its $q \to -q$ partner have identically vanishing coefficients.

The (Roman numeral) collections in the tables/graphs that contain just two eta quotients consist of an eta quotient and its $q \to -q$ partner.

Recap I

Recall:

Table 2: Eta quotients with vanishing behaviour similar to f_1^4

Collection	# of eta quotients	Collection	# of eta quotients
I	72	II *	4
III †	2	IV	6
V †	2	VI *	4
VII *	6	VIII *	8
IX *	4	X	4
XI	14	XII †	2
XIII †	2	XIV †	2
XV	4	XVI †	2
XVII	4	XVIII †	2
XIX *	6		

Recap II

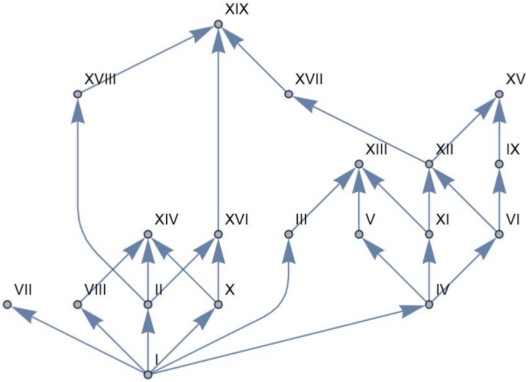

Figure: The grouping of the 150 eta-quotients in Table 1, which have vanishing coefficient behaviour similar to f_1^4

Recap III

As mentioned previously, we showed that if $A(q)=f_1^4$ and $B(q)$ is any one of the 150 eta quotients in the table/graph, then

$$A_{(0)} \subseteq B_{(0)}.$$

However most of the "fine structure" of the tables/graphs (identical vanishing of coefficients for all eta quotients in each collection, and strict inclusion between sets of vanishing coefficients for any pair of eta quotients in two different collections joined by a line segment in a graph) was not proven.

We next describe a method that allows some of this fine structure to be proven.

The m-Dissection of a Function,I

Definition

By the *m-dissection* of a function $G(q)=\sum_{n=0}^{\infty} g_n q^n$ we mean an expansion of the form

$$G(q)=\gamma_0 G_0(q^m)+\gamma_1 q G_1(q^m)+\cdots+\gamma_{m-1} q^{m-1} G_{m-1}(q^m), \tag{19}$$

where each dissection component $G_i(q^m)$ is not identically zero ($\gamma_i=0$ is allowed). In other words, for each i, $0\leq i\leq m-1$,

$$\gamma_i\, q^i\, G_i(q^m)=\sum_{n=0}^{\infty} g_{mn+i} q^{mn+i}=q^i \sum_{n=0}^{\infty} g_{mn+i} (q^m)^n.$$

Similar m-Dissections

Now suppose $C(q)$ and $D(q)$ are two functions whose m-dissections are given by

$$
\begin{aligned}
C(q) &= c_0 G_0(q^m) + c_1 q G_1(q^m) + \cdots + c_{m-1} q^{m-1} G_{m-1}(q^m), \qquad (20)\\
D(q) &= d_0 G_0(q^m) + d_1 q G_1(q^m) + \cdots + d_{m-1} q^{m-1} G_{m-1}(q^m).
\end{aligned}
$$

There are two cases of interest.

1) Suppose that $c_i = 0 \iff d_i = 0$, $i = 0, 1, \ldots, m-1$, and then it is clear that $C_{(0)} = D_{(0)}$.

If the c_1, d_i satisfy the condition just stated, we say that $C(q)$ and $D(q)$ have *similar* m-dissections.

2) On the other hand, if $c_j \neq 0$ and $d_j = 0$ for one or more $j \in \{0, 1, \ldots, m-1\}$ and otherwise $c_i = 0 \iff d_i = 0$, then $C_{(0)} \subsetneqq D_{(0)}$.

The following 2-dissection identities are well known:

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8}, \tag{21}$$

$$\frac{f_1^2}{f_2} = \frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}, \tag{22}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{23}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^3 f_6^3}{f_2^9 f_{12}}\left(\frac{f_4^3}{f_{12}} + 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right), \tag{24}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4} \implies \frac{f_4^3 f_6}{f_2^7 f_{12}}\,\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^3}{f_2^9 f_{12}}\left(\frac{f_4^3}{f_{12}} + q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right), \tag{25}$$

$$\frac{f_1}{f_3^3} = \frac{f_2^3 f_{12}^3}{f_4 f_6^9}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} - q\frac{f_{12}^3}{f_4}\right), \tag{26}$$

Some 2-Dissections, II

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{27}$$

$$\frac{1}{f_1 f_3} = \frac{f_4 f_{12}}{f_2^3 f_6^3}\left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} + q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}\right), \tag{28}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{29}$$

$$\frac{1}{f_1^4} = \frac{f_4^4}{f_2^{12}}\left(\frac{f_4^{10}}{f_2^2 f_8^4} + 4q\frac{f_2^2 f_8^4}{f_4^2}\right), \tag{30}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}, \tag{31}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6^3}{f_2^3 f_{12}}\left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} + q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}\right), \tag{32}$$

Some 2-Dissections, III

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}, \tag{33}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^2 f_6^6}{f_2^6 f_{12}^2}\left(\frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} + 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}\right). \tag{34}$$

The 2-dissections mentioned above (in red), and their $q \to -q$ partners, give the vanishing coefficient result in the next theorem.

A Theorem on Identical Vanishing of Coefficients

Theorem

Let $C(q^2)$ be any even eta quotient. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:

$$\left\{ \frac{f_3}{f_1^3}C(q^2),\ \frac{f_1^3 f_4^3 f_6^3}{f_2^9 f_3 f_{12}}C(q^2),\ \frac{f_3^3}{f_1}\frac{f_4^3 f_6}{f_2^7 f_{12}}C(q^2),\ \frac{f_1}{f_3^3}\frac{f_4^4 f_6^{10}}{f_2^{10} f_{12}^4}C(q^2) \right\}. \tag{35}$$

Then

$$F_{(0)} = G_{(0)}. \tag{36}$$

Specializing $C(q^2)$ then shows that various collections of 4 eta quotients in some of the tables have identically vanishing coefficients.

The following 3-dissections are also well known:

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{37}$$

$$\frac{f_1 f_4}{f_2} = \frac{f_3 f_{12} f_{18}^5}{f_6^2 f_9^2 f_{36}^2} - q\frac{f_9 f_{36}}{f_{18}}, \tag{38}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \Longrightarrow \frac{f_6}{f_3}\frac{f_1^2}{f_2} = \frac{f_6 f_9^2}{f_3 f_{18}} - 2q\frac{f_{18}^2}{f_9} \tag{39}$$

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_{18}^5}{f_9^2 f_{36}^2} + \frac{2q f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}, \tag{40}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{41}$$

$$\frac{f_1^2 f_4^2}{f_2^5} = \frac{f_3^8 f_{12}^8 f_{18}^{15}}{f_6^{20} f_9^6 f_{36}^6} - \frac{2q f_3^7 f_{12}^7 f_{18}^9}{f_6^{18} f_9^3 f_{36}^3} + \frac{4q^2 f_3^6 f_{12}^6 f_{18}^3}{f_6^{16}}. \tag{42}$$

The Borwein Theta Functions

Recall that the Borwein theta functions $a(q)$, $b(q)$ and $c(q)$ are defined by

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = \frac{f_2^5 f_6^5}{f_1^2 f_3^2 f_4^2 f_{12}^2} + 4q\frac{f_4^2 f_{12}^2}{f_2 f_6}, \tag{43}$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{m^2+mn+n^2} = \frac{f_1^3}{f_3},$$

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2} = 3q^{1/3}\frac{f_3^3}{f_1},$$

where $\omega = \exp(2\pi i/3)$.
Aside: The functions above satisfy the identity

$$a(q)^3 = b(q)^3 + c(q)^3.$$

Some 3-Dissections, I

Lemma

The following 3-dissections hold.

$$f_1^3 = a(q^3)f_3 - 3qf_9^3 \Longrightarrow f_1^6 = f_3^2\left(a(q^3)^2 - 6q\frac{f_9^3}{f_3}a(q^3) + 9q^2\frac{f_9^6}{f_3^2}\right) \tag{44}$$

$$\frac{1}{f_1^3} = \frac{f_9^3}{f_3^{10}}\left(a(q^3)^2 + 3q\frac{f_9^3}{f_3}a(q^3) + 9q^2\frac{f_9^6}{f_3^2}\right) \Longrightarrow$$
$$\frac{f_3^{12}}{f_9^3 f_1^3} = f_3^2\left(a(q^3)^2 + 3q\frac{f_9^3}{f_3}a(q^3) + 9q^2\frac{f_9^6}{f_3^2}\right). \tag{45}$$

More Vanishing Coefficient Results, I

Theorem

Let $C(q^3)$ be any eta quotient whose series expansion contains only powers of q^3. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following lists:

$$\left\{ \frac{f_2^2}{f_1} C(q^3), \frac{f_1 f_4}{f_2} C(-q^3), \frac{f_1^2 f_6}{f_2 f_3} C(q^3), \frac{f_2^5 f_3 f_{12}}{f_1^2 f_4^2 f_6^2} C(-q^3) \right\}, \tag{46}$$

$$\left\{ \frac{f_1^2 f_8}{f_4} C(q^3), \frac{f_2^6 f_8}{f_1^2 f_4^3} C(-q^3), \frac{f_3 f_4^5 f_{24}}{f_1 f_8^2 f_{12}^2} C(q^3), \frac{f_1 f_4^6 f_6^3 f_{24}}{f_2^3 f_3 f_8^2 f_{12}^3} C(-q^3) \right\}, \tag{47}$$

$$\left\{ f_1^6 C(q^3), \frac{f_2^{18}}{f_1^6 f_4^6} C(-q^3), \frac{f_3^{12}}{f_1^3 f_9^3} C(q^3), \frac{f_1^3 f_4^3 f_6^{36} f_9^3 f_{36}^3}{f_2^9 f_3^{12} f_{12}^{12} f_{18}^9} C(-q^3) \right\}, \tag{48}$$

Then

$$F_{(0)} = G_{(0)}. \tag{49}$$

More Vanishing Coefficient Results, II

As with the previous theorem, here also Specializing $C(q^3)$ then shows that various collections of 4 eta quotients in some of the tables have identically vanishing coefficients.

Some 4-Dissections, I

We give an example of one of the ways more complicated dissection results are obtained by combining the basic dissection results in various ways. Recall

$$\frac{f_1^2}{f_2} = \frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}, \tag{50}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{51}$$

Note that (50) actually gives a 4-dissection of f_1^2/f_2.

We will use the second identity with $q \to q^2$, giving a 4-dissection of f_2^4.

Some 4-Dissections, II

Lemma

The following 4-dissections hold.

$$f_1^2 f_2^7 = \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)^2, \tag{52}$$

$$\frac{1}{f_1^2 f_2^3} = \frac{f_8^8}{f_4^{22}}\left(\frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right)^2. \tag{53}$$

Proof.

For (52), write

$$f_1^2 f_2^7 = \frac{f_1^2}{f_2}(f_2^4)^2$$

and use (50) and (51), with q replaced with q^2 in the latter identity. The proof of (53) is similar. □

Some 4-Dissections, III

Observe that $f_1^2 f_2^7$ and $f_4^{22}/(f_1^2 f_2^3 f_8^8)$ have similar 4-dissections.

Theorem

Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:

$$\left\{ f_1^2 f_2^7 C(q^4),\ \frac{f_2^{13}}{f_1^2 f_4^2} C(q^4),\ \frac{1}{f_1^2 f_2^3}\frac{f_4^{22}}{f_8^8} C(q^4),\ \frac{f_1^2}{f_2^9}\frac{f_4^{24}}{f_8^8} C(q^4) \right\}. \tag{54}$$

Then

$$F_{(0)} = G_{(0)}. \tag{55}$$

Remark: All of the m-dissection derived were derived to prove identically vanishing coefficient results.

More New m-Dissection Results, I

All of the dissections in the next lemma were derived by combining the "basic" (well known) 2- and 3- dissections in various ways.

More New m-Dissection Results, II

Lemma

The following 4-dissections hold:

$$\frac{f_2}{f_1^2} = \frac{f_8^4}{f_4^{10}}\left(\frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} + 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right), \tag{56}$$

$$f_1^2 f_2^3 = \left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_8^{10}}{f_4^2 f_{16}^4} - 4q^2\frac{f_4^2 f_{16}^4}{f_8^2}\right) \tag{57}$$

$$= \frac{f_8^{15}}{f_4^4 f_{16}^6} - \frac{2qf_8^9}{f_4^2 f_{16}^2} - 4q^2 f_8^3 f_{16}^2 + \frac{8q^3 f_4^2 f_{16}^6}{f_8^3}, \tag{58}$$

$$\frac{f_1^6}{f_2^3} = \frac{f_8^{15}}{f_4^6 f_{16}^6} - 6q\frac{f_8^9}{f_4^4 f_{16}^2} + 12q^2\frac{f_8^3 f_{16}^2}{f_4^2} - 8q^3\frac{f_{16}^6}{f_8^3}. \tag{59}$$

Notice that f_2/f_1^2, $f_1^2 f_2^3(f_8^4/f_4^{10})$ and $f_1^6 f_8^4/f_2^3 f_4^8$ have similar 4-dissections, so that if each of these is multiplied by any eta quotient $C(q^4)$, the resulting eta quotients will have identically vanishing coefficients.

More New m-Dissection Results, III

Theorem

Let $C(q^4)$ be any eta quotient whose series expansion contains only powers of q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients in the following list:

$$\left\{ \frac{f_2}{f_1^2}C(q^4),\ \frac{f_1^2 f_4^2}{f_2^5}C(q^4),\ \frac{f_1^2 f_2^3 f_8^4}{f_4^{10}}C(q^4),\right.$$
$$\left. \frac{f_2^9 f_8^4}{f_1^2 f_4^{12}}C(q^4),\ \frac{f_1^6 f_8^4}{f_2^3 f_4^8}C(q^4),\ \frac{f_2^{15} f_8^4}{f_1^6 f_4^{14}}C(q^4)\right\}. \tag{60}$$

Then

$$F_{(0)} = G_{(0)}. \tag{61}$$

More New m-Dissection Results, IV

We next consider a new type of dissection result, one in which the components of the dissections are not just simple eta quotients.

We need the lemma in the next slide.

We recall the notation, for a an integer and m a positive integer,

$$\bar{J}_{a,m} := (-q^a, -q^{m-a}, q^m; q^m)_\infty. \tag{62}$$

More New m-Dissection Results, V

Lemma

The following 2-dissections hold.

$$f_1 = \frac{f_2}{f_4}\left(\bar{J}_{6,16} - q\bar{J}_{2,16}\right), \tag{63}$$

$$\frac{1}{f_1} = \frac{1}{f_2^2}\left(\bar{J}_{6,16} + q\bar{J}_{2,16}\right). \tag{64}$$

Proof.

The identity (64) was proven by Hirschhorn, and (63) is its $q \to -q$ partner. □

In the paper we proved 15 pairs of 4-dissections by combining the dissections above with the basic 2- and 3- dissections in ways similar to what has been seen already.

The following 4-dissections hold.

$$\frac{f_1^2}{f_2^2} = \frac{1}{f_4^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{12,32} + q^2 \bar{J}_{4,32}\right), \tag{65}$$

$$f_1^2 = \frac{f_4}{f_8}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\bar{J}_{12,32} - q^2 \bar{J}_{4,32}\right), \tag{66}$$

Note that

$$f_1^2 \text{ and } \frac{f_1^2 f_4^3}{f_2^2 f_8}$$

have similar 4-dissections.

More New m-Dissection Results, XVI

As a consequence of the dissections on the previous slide:
Theorem. *Let $C(q^4)$ be any eta quotient with a power series expansion in q^4. Let $F(q)$ and $G(q)$ be any pair of eta quotients from one the following list:*

$$\left\{ f_1^2 C\left(q^4\right), \frac{f_2^6}{f_1^2 f_4^2} C\left(q^4\right), \frac{f_1^2 f_4^3}{f_2^2 f_8} C\left(q^4\right), \frac{f_2^4 f_4}{f_1^2 f_8} C\left(q^4\right) \right\}. \tag{67}$$

Then

$$F_{(0)} = G_{(0)}. \tag{68}$$

Remark: The other 14 pairs of 4-dissections lead to similar vanishing coefficient results.

More New m-Dissection Results, XIX

Now $C(q^4)$ can be specialized to prove vanishing coefficient results for collections of eta quotients which experiment indicated had vanishing coefficient similar to that of one of f_1^r, $r = 4, 6, 8, 10, 14$ and 26 or $f_1^3 f_2^3$.

Similar reasoning also leads to strict inclusion results.

Together, these allow some of the “fine structure” of the tables/graphs to be proven.

We close this section with two examples.

A Collection of Eta Quotients with Identically Vanishing Coefficients

Let $F(q)$ and $G(q)$ be any two eta quotients from the following collection (which is from the table/graph for f_1^4):

$$\left\{ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \right.$$
$$\left. \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}} \right\}.$$

Then

$$F_{(0)} = G_{(0)}.$$

An Example of Strict Inclusion of Sets of Vanishing Coefficients

The following pair of collections of eta quotients are also from the table/graph for f_1^4 (actually VIII is the collection in the previous example) :

$$VIII = \left\{ \frac{f_2^3 f_3 f_8 f_{12}^8}{f_1 f_4^3 f_6^4 f_{24}^3}, \frac{f_1 f_8 f_{12}^7}{f_3 f_4^2 f_6 f_{24}^3}, \frac{f_1 f_4^8 f_6^3 f_{24}}{f_2^4 f_3 f_8^3 f_{12}^3}, \frac{f_3 f_4^7 f_{24}}{f_1 f_2 f_8^3 f_{12}^2}, \frac{f_1 f_2^2 f_6}{f_3 f_4}, \frac{f_2^5 f_3 f_{12}}{f_1 f_4^2 f_6^2}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3 f_{12}^2}, \frac{f_2 f_3 f_6^2}{f_1 f_{12}} \right\},$$

$$XIV = \left\{ \frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}, \frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}} \right\}.$$

If $A(q)$ is any of the 8 eta quotients in collection VIII and $B(q)$ is either of the 2 eta quotients in collection XIV, then

$$A_{(0)} \subsetneq B_{(0)}.$$

Applications to Proving Results from the Various Tables and Graphs

We illustrate we have managed to prove using the table/graph for f_1^6.

In what follows, anything coloured *red* indicates a result proved by the dissection methods described.

Anything coloured *green* is either trivially true (because a Roman-numeral collection contains just two eta-quotients that are $q \to -q$ partners of each other) or else derives from an already known result through a $q \to q^k$ dilation (some integer $k > 1$).

There are also many partial results not shown.

The Case of f_1^6 I

Table 3: Eta quotients with vanishing behaviour similar to f_1^6

Collection	# of eta quotients	Collection	# of eta quotients
I	42	II	4
III	4	IV	16
V	2	VI	2
VII	4	VIII	4
IX	4	X	10
XI	2	XII	4
XIII	8	XIV	4
XV	8	XVI	2
XVII	8	XVIII	2
XIX	2	XX	2
XXI	4	XXII	6
XXIII	2	XXIV	4
XXV	4	XXVI	4
XXVII	2	XXVIII	6
XXIX	6		

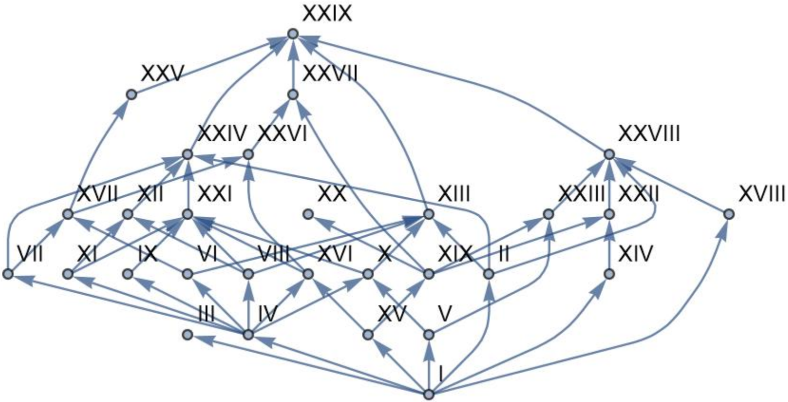

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^6, according to experimental evidence

The Case of f_1^6 III

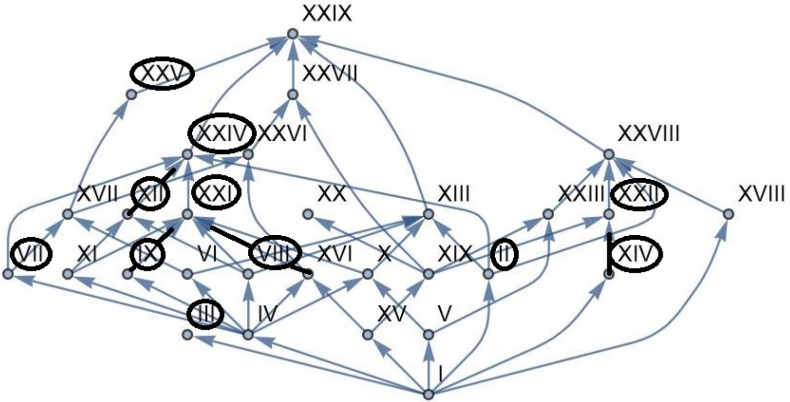

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^6 - what has been proven

The Case of f_1^6 IV

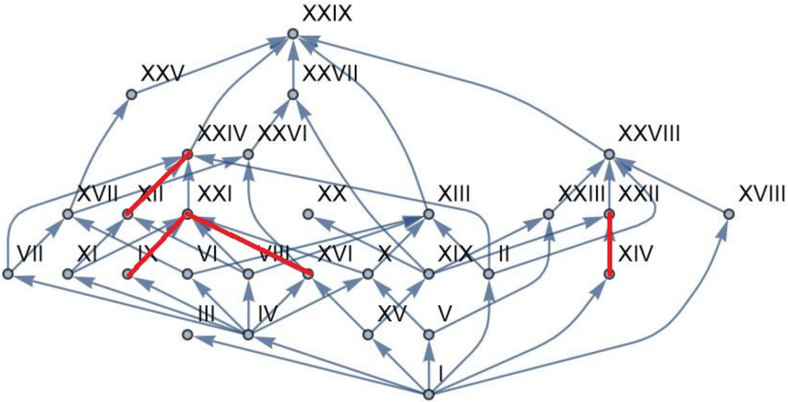

Figure: The grouping of eta-quotients in Table 3, which have vanishing coefficient behaviour similar to f_1^6, proven inclusion results

Open Questions, Conjectures, Further Directions

A Sturm-type Bound for Identically Vanishing Coefficients?

Open Problem

Fix a weight k and a level ℓ. Is it true that there is always a positive integer $N = N(k, \ell)$ such that for any holomorphic modular forms $\sum_{n=0}^{\infty} a_n q^n$ and $\sum_{n=0}^{\infty} b_n q^n$ of weight k and level $\Gamma_0(\ell)$, whenever

$$\{n \leq N \,|\, a_n = 0\} = \{n \leq N \,|\, b_n = 0\},$$

then

$$\{n \geq 0 \,|\, a_n = 0\} = \{n \geq 0 \,|\, b_n = 0\}?$$

A Restriction on the Pairs of Eta Quotients $(A(q), B(q))$ for which $A_{(0)} = B_{(0)}$?

Open Problem

Let $A(q) = \prod_j f_j^{n_j}$, $B(q) = \prod_i f_i^{n_i}$ *be two eta quotients such that* $A_{(0)} = B_{(0)}$.
Is it necessarily the case that

$$\sum_j j n_j = \sum_i i n_i ?$$

For example, consider the following subset of eta quotients whose coefficients vanish identically with those of f_1^4*:*

$$\frac{f_1^2 f_4^2 f_6^5}{f_2^3 f_3^2 f_{12}^2}, \frac{f_2^6 f_3^2}{f_1^2 f_6^2}, \frac{f_1^2 f_4^2 f_6^4}{f_3^2 f_{12}^2}, \frac{f_2^7 f_6^5}{f_1^2 f_3^2 f_4^2 f_{12}^2}, \frac{f_1^2 f_2 f_3^2}{f_6}, \frac{f_2^9}{f_1^2 f_4^3}, \frac{f_1^2 f_2^3}{f_4}.$$

It is seen that in each case $\sum_j j n_j = 4$.

How Extensive are the Phenomena?

For example,

- How many eta quotients in total have coefficients that vanish identically with f_1^4? (There appears to be 72 at present, from a limited search.) Finitely many? Infinitely many?

- How far "upwards" does the tree of eta quotients partially ordered by inclusion extend, both in terms of total number of eta quotients (presently 150) and in terms of branches (presently 19 distinct collections)? Is the extent finite or infinite in either sense?

Similar questions can be asked about any of the other families of lacunary eta quotients considered.

The Existence of “Isolated” Lacunary Eta Quotients?

Does every lacunary eta quotient exist (in terms of vanishing coefficients) somewhere in some tree-like structure such as was shown for f_1^4 and f_1^6?

Alternatively, are there “isolated” lacunary eta quotients with no connections (in terms of vanishing coefficients) with any other eta quotients (apart from its $q \to -q$ partner)?

Does the Phenomenon Extend "Downwards"

Does the Phenomenon Extend "Downwards"?

For example, are there lacunary eta quotients $A(q)$ such that if $B(q) = f_1^4$,

$$A_{(0)} \subsetneqq B_{(0)}?$$

How extensive might this phenomenon be, if it exists?

Some interesting phenomena were observed experimentally.

Why are the Phenomena so sparse for f_1, f_1^2 and f_1^3

Our searches found, respectively, just 6, 10 and 12 eta quotients with vanishing coefficient behaviour similar to, respectively, f_1, f_1^2 and f_1^3.

In contrast, there were 150 eta quotients in the case of f_1^4, and similar numbers (from 88 to 172) for f_1^6, f_1^8, f_1^{10}, f_1^{14}, f_1^{26} and $f_1^3 f_2^3$.

Why are the numbers much smaller for f_1, f_1^2 and f_1^3?

In the cases of f_1 and f_1^3, it might be because their series expansions are *superlacunary*, but if so, what is the explanation in the case of f_1^2?

In the case of all three (f_1, f_1^2 and f_1^3) would more extensive searches turn up other eta quotients with similar vanishing coefficient behaviour, or are the stated lists complete?

Vanishing Coefficient Series Results from Theta Series Expansions?

As mentioned above, a good many of the lacunary eta quotients in the table may be expressed as products of eta-quotients with single sum theta series expansions. For example,

$$\frac{\eta(12z)^3\eta(18z)^2}{\eta(6z)\eta(36z)} = \sum_{m,n\geq 1}\left(\frac{m}{2}\right)^2\left(\frac{n}{6}\right)^2 q^{\frac{1}{4}\left(3m^2+n^2\right)} =: \sum_{t=1}^{\infty} a_t^* q^t,$$

$$\frac{\eta(12z)^{13}}{\eta(6z)\eta(24z)^5} = \sum_{m,n\geq 1} mn\left(\frac{-4}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}\left(3m^2+n^2\right)} =: \sum_{t=1}^{\infty} b_t^* q^t.$$

Experiment suggests

$$a_t^* = 0 \iff b_t^* = 0.$$

Can this be proven by considering the coefficients in the double theta series?

There are many instances in the various tables and graphs.

Some References, Oldest to Newest I

- Serre, J.-P. *Sur la lacunarité des puissances de η.* Glasgow Math. J. **27** (1985), 203–221.
- Ewell, John A. *On sums of triangular numbers and sums of squares.* Amer. Math. Monthly **99** (1992), no. 8, 752–757.
- Han, Guo-Niu; Ono, Ken *Hook lengths and 3-cores.* Ann. Comb. **15** (2011), no. 2, 305–312.
- Huber, T.; McLaughlin, J.; Ye, D. *Lacunary eta quotients with identically vanishing coefficients* , Int. J. Number Theory **19** (2023), 1639–1670.
- Huber, T.; Liu, CC.; McLaughlin, J.; Ye, D.; Yuan, M.; Zhang, S. On the vanishing of the coefficients of CM eta quotients - to appear in Proceedings of the Edinburgh Mathematical Society.

Some References, Oldest to Newest II

Huber, T.; Mc Laughlin, J.; Ye, D. *Further Results on Vanishing Coefficients in the Series Expansion of Lacunary Eta Quotients* - submitted.

Huber, T.; Mc Laughlin, J.; Ye, D. *Dissections of lacunary eta quotients and identically vanishing coefficients* - submitted.

Thanks

Thank you for listening/watching.