ON PAIRS OF DIAGONAL QUINTIC FORMS

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1. Introduction

The application of the Hardy-Littlewood method to simultaneous diagonal equations provides a rare instance, in the investigation of diophantine equations, in which reasonable bounds may be established for the number of variables required to guarantee the existence of non-trivial integral solutions, subject only to local solubility conditions. Beginning with work of Davenport and Lewis in the 1960s (see [9], [10]), workers have sought to exploit developments in the circle method to reap improved conclusions for simultaneous diagonal equations, and especially for pairs of such equations (see in particular [2]–[7] and [12]). Oftentimes serious technical complications are encountered in such endeavours, and this discourages widespread use of the new tools. Most recently, Brüdern [3] has investigated pairs of diagonal cubic equations in 14 variables, developing a p-adic iteration restricted to minor arcs appropriate to this problem. Formidable technical difficulties permeate the latter treatment, and the length and complexity of the associated exposition apparently deterred application of this method to a related problem involving pairs of diagonal quintic equations (see the preamble to Theorem 3 of Brüdern [4]).

The purpose of this paper is two-fold. On the one hand, we exploit recent mean value estimates for smooth Weyl sums due to Vaughan and Wooley [17] in an investigation of the solubility of pairs of diagonal quintic equations. On the other hand, we seek to provide a flexible alternative to the technically burdensome methods based on p-adic iterations restricted to minor arcs. Our approach would appear to possess the same power as that potentially attainable via the latter methods, yet is sufficiently simple that workers might be tempted to apply it to related problems. In order to be more specific concerning the central problem of this paper, let $c_1, \dots, c_s$ and $d_1, \dots, d_s$ be integers, and consider the system of equations

$$c_1 x_1^5 + \cdots + c_s x_s^5 = d_1 x_1^5 + \cdots + d_s x_s^5 = 0. \tag{1.1}$$

We seek to determine how large s must be to ensure the existence of a non-trivial integral solution $\mathbf{x}$ to this system (that is, a solution $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$).

Theorem 1. *Suppose that $s \geq 34$, and that the system* (1.1) *possesses a non-trivial* 11-*adic solution. Then the pair of equations* (1.1) *possesses a non-trivial integral solution.*

One of the first results of this type was obtained by Cook [7], who established without any local solubility hypothesis that the system (1.1) has non-trivial integral solutions whenever $s \geq 51$. Assuming the 11-adic solubility of the system (1.1), Brüdern [4] (see

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Theorem 3) improved this bound to $s \geq 37$ by employing mean value estimates of Vaughan [14]. Although the latter bound could be further sharpened to $s \geq 35$ by routinely exploiting recent work of Vaughan and Wooley [17] concerning Waring's problem for fifth powers, the conclusion embodied in Theorem 1 requires an altogether more sophisticated strategy, and appears to be the best attainable in the current state of technology. We remark that Cook [8] has shown that the system (1.1) has non-trivial 11-adic solutions whenever $s \geq 41$, and has also shown that when $p \neq 11$, the existence of non-trivial p-adic solutions is assured whenever $s \geq 31$. Finally, as is mentioned in Atkinson and Cook [1], it is a simple matter to construct examples of the type (1.1) with $s = 30$ that fail to possess non-trivial 11-adic solutions.

In the crudest approaches to problems of the type discussed above, estimates of Weyl-type for exponential sums compensate for deficiencies in the available mean value estimates, and hence permit an application of the Hardy-Littlewood method via a traditional division into major and minor arcs. When such an approach fails, recent "efficient differencing" methods for estimating mean values of exponential sums over complete unit intervals sometimes fail to establish the desired conclusion by only the narrowest of margins. Efficient differencing may nonetheless be attempted in such situations, but now one seeks to restrict the mean value to a set of minor arcs. First applied by Vaughan [13] in work on sums of cubes, it is this approach that Brüdern applies in his impressive tour-de-force [3] devoted to pairs of diagonal cubic equations. Such a strategy entails laboriously tracking the whereabouts of the minor arcs through myriad changes of variable, and requires an arsenal of precise estimates essential for delicate pruning analyses. Instead of diving into the technical morass of differencing on minor arcs, we exploit the marginal failure of conventional efficient differencing through an alternative mechanism. Given a pair of equations (1.1) in 34 variables, we apply an efficient differencing process to a mean value, over the complete unit square, involving a 33.998th moment of quintic exponential sums. Drawing inspiration from work of Wooley [19] devoted to fractional moments of smooth Weyl sums, this task proves to be relatively straightforward. Our estimate for this mean value scarcely misses the expected (best possible) upper bound, and moreover a small fraction of an exponential sum remains with which to gain additional cancellation on the minor arcs. Indeed, equipped with the fractional moment just alluded to (see Theorem 2.1 below), it now suffices to apply a crude approach similar to that discussed earlier. We emphasise here the overwhelming simplifications achieved by integrating over a complete unit square in the differencing process, and those achieved by aiming for a slightly imprecise upper bound, over the corresponding complex and delicate analysis required by restricting oneself to minor arcs (as in Brüdern [3]).

We begin by establishing our fundamental mean value estimate in §2 using the ideas alluded to above. In §3, we make some simplifying reductions and then describe our approach to the theorem via the Hardy-Littlewood method. With the mean value estimate of §2 in hand, we are able to deal with the minor arcs in a routine manner in §4. As we are forced to handle a relatively thick set of major arcs, the pruning operation undertaken in §5 is not without its technical hurdles. Yet, once these obstacles have been negotiated, we are able to perform the usual end-game analysis in §6 with few unexpected difficulties.

Throughout, the letter ε denotes a sufficiently small positive number. We take P to be the basic parameter, and this is always presumed sufficiently large in terms of ε. The implicit constants in Vinogradov's well-known notation, $\ll$ and $\gg$, depend on ε and the coefficients of implicit diophantine equations, unless otherwise indicated. When π is a prime number, we write $\pi^h \| n$ to denote that $\pi^h | n$ but $\pi^{h+1} \nmid n$. Finally, we adopt the convention that whenever ε appears in a statement, either implicitly or explicitly, then we assert that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε.

2. An Auxiliary Mean Value Estimate

The relative ease with which we establish the central conclusion of this paper is a consequence of a sharp estimate for a certain fractional moment of quintic Weyl sums. By employing an efficient differencing process related to that applied in Wooley [19], we are able to difference four complete Weyl sums in a mean value including somewhat fewer than 34 exponential sums. Before describing our conclusions in detail, we require some notation.

Consider fixed integers A, B, a, b, c and d with the property that $A > 0$, $B > 0$, $ad \neq 0$ and $ad - bc \neq 0$. We take η to be a small positive number, and P to be a positive number sufficiently large in terms of η, A, B, a, b, c and d. Write

$$M = P^{7/41}, \quad Q = PM^{-1}, \quad H = PM^{-5} \quad \text{and} \quad R = P^{\eta}. \tag{2.1}$$

As usual, we write $e(z)$ for $e^{2\pi i z}$. When k is a natural number, we write

$$F_k(\theta) = \sum_{\substack{1 \le x \le P \\ (x,k)=1}} e(\theta x^5). \tag{2.2}$$

Also, when X and Y are positive numbers, we define the set of Y-smooth numbers not exceeding X by

$$\mathcal{A}(X, Y) = \{n \in [1, X] \cap \mathbb{Z} : p | n \Rightarrow p \le Y\},$$

and then define

$$f(\theta) = \sum_{y \in \mathcal{A}(Q,R)} e(\theta y^5). \tag{2.3}$$

When $(\alpha, \beta) \in [0, 1]^2$, it is convenient to define $\Lambda_1 = \Lambda_1(\alpha, \beta)$ and $\Lambda_2 = \Lambda_2(\alpha, \beta)$ by $\Lambda_1 = a\alpha + b\beta$ and $\Lambda_2 = c\alpha + d\beta$. Finally, for $0 \le t \le 1$, we define the exponential sum

$$\mathcal{F}_t(\alpha, \beta) = \sum_{\substack{M < p \le 2M \\ p \equiv -1 \pmod 5}} |F_p(A\alpha) F_p(B\beta)|^2 |f(\Lambda_1 p^5) f(\Lambda_2 p^5)|^{15-t}, \tag{2.4}$$

where here and throughout, the letter p denotes a prime number.

Our objective in this section is the proof of the estimate contained in the following theorem.

Theorem 2.1. *Suppose that t is a real number with $0 \le t \le 10^{-3}$. Then whenever $\eta > 0$ is sufficiently small, one has for each positive number ε the estimate*

$$\iint_{[0,1]^2} \mathcal{F}_t(\alpha, \beta) \, d\alpha \, d\beta \ll M P^{\varepsilon - 6} Q^{30 - 2t}.$$

Before launching our proof of Theorem 2.1, it is useful to establish some preliminary estimates that ease our subsequent discussion. We begin by recalling some mean value estimates of Vaughan and Wooley [17].

Lemma 2.2. *When* $s = 7, 8, 9$, *one has*

$$\int_0^1 |f(\theta)|^{2s}\, d\theta \ll Q^{2s-5+\Delta_s},$$

where $\Delta_7 = 0.272729$, $\Delta_8 = 0.077363$ *and* $\Delta_9 = 0$.

Proof. These estimates are immediate from the tables on page 236 of [17]. □

Following the execution of our differencing procedure, we obtain the exponential sum

$$\mathcal{G}_1(\theta; p; C) = p^{-5} \sum_{\ell=1}^{p^5} \left| F_p \left(\frac{C(\theta + \ell)}{p^5} \right) \right|^2. \tag{2.5}$$

We require an estimate for the related exponential sum

$$F_1^*(\theta) = \sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} |\mathcal{G}_1(\theta; p; C)|^2, \tag{2.6}$$

valid for $C = A$ or B, and valid uniformly for $\theta \in [0, 1)$. In order to describe this estimate, we require some further notation. We put $c = 10^6 \max\{A, B\}$ and then define the set of major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q, r) = \{\theta \in [0, 1) : |q\theta - r| \le c^{-1} P Q^{-5}\},$$

with $0 \le r \le q \le c^{-1}P$ and $(r, q) = 1$. We also set $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. We write

$$\Phi(z, h, p) = p^{-5}((z + hp^5)^5 - (z - hp^5)^5),$$

and then define

$$\tau(q, r, h, p) = \left| \sum_{w=1}^{q} e\left(\frac{r}{q} \Phi(w, h, p) \right) \right|.$$

Also, we define the function $\Delta_C(\theta)$ for $\theta \in [0, 1)$ by taking

$$\Delta_C(\theta) = \sum_{M<p\le 2M} \left(\sum_{1\le h\le H} \frac{Pq^{-1}\tau(q, Cr, h, p)}{(1 + |\theta - r/q|hP^4)^{1/4}} \right)^2,$$

when $\theta \in \mathfrak{M}(q, r) \subseteq \mathfrak{M}$, and otherwise by taking $\Delta_C(\theta) = 0$.

Lemma 2.3. *When* $C = A$ *or* B, *the estimate* $F_1^*(\theta) \ll P^{2+\varepsilon} M + \Delta_C(\theta)$ *holds uniformly for* $\theta \in [0, 1)$.

Proof. Suppose that C is either A or B, and let p be a prime number satisfying $M < p \le 2M$ and $p \equiv -1 \pmod 5$. Then in view of (2.1), and our assumption that P is sufficiently large in terms of A and B, one has $p > \max\{A, B\}$, whence p does not divide AB. Then we may apply the argument on page 46 of Vaughan and Wooley [16], leading to equation (4.14) of that paper, to conclude that

$$|\mathcal{G}_1(\theta; p; C)| \ll P + |G_p(C\theta)|, \tag{2.7}$$

where

$$G_p(\xi)=\sum_{1\le h\le Pp^{-5}}\sum_{\substack{hp^5<z\le 2P-hp^5\\ z\equiv h \pmod 2}} e(2^{-5}\xi\Phi(z,h,p)). \tag{2.8}$$

Further, on applying Lemma 4.1 of Vaughan and Wooley [16], we derive the estimate

$$|\mathcal{G}_1(\theta;p;C)|\ll P+(\log P)G_p^*(C\theta), \tag{2.9}$$

where

$$G_p^*(\xi)=\sum_{1\le h\le H}\sup_{\gamma\in[0,1]}\left|\sum_{1\le z\le 2P} e(2^{-5}\xi\Phi(z,h,p)+\gamma z)\right|. \tag{2.10}$$

We first obtain an estimate of major arc type for the exponential sum defined in (2.8). By applying essentially the same van der Corput analysis as was used in the proof of Lemma 4.7 of Vaughan and Wooley [17], one finds that when $C=A$ or B, and $\theta\in\mathfrak{M}(q,r)\subseteq\mathfrak{M}$, one has

$$|G_p(C\theta)|\ll\sum_{1\le h\le H}\frac{Pq^{-1}\tau(q,Cr,h,p)}{(1+|\theta-r/q|hP^4)^{1/4}}+Hq^{3/4+\varepsilon}. \tag{2.11}$$

Here we note that the restrictions on the variable z imposed in (2.8) are easily accommodated within the latter argument (the reader may wish to compare the situation here with that in the proof of Lemma 4.3 of Vaughan and Wooley [16]). On recalling (2.7), we therefore conclude from (2.1), (2.6) and (2.11) that whenever $\theta\in\mathfrak{M}$ one has

$$F_1^*(\theta)\ll\Delta_C(\theta)+M(P+HP^{3/4+\varepsilon})^2\ll\Delta_C(\theta)+P^2M. \tag{2.12}$$

We next observe that one may treat the exponential sum

$$F_1^+(\theta)=\sum_{M<p\le 2M}|G_p^*(C\theta)|^2 \tag{2.13}$$

by using a refined differencing argument similar to that applied in §§2 and 3 of Vaughan [14] to the exponential sum

$$F_1(\theta)=\sum_{M<m\le MR}\sum_{1\le h\le H}\sum_{1\le z\le 2P} e(\theta\Phi(z,h,m)).$$

Following an application of Cauchy's inequality, the differencing process removes the implicit supremum over γ in (2.10) and hence gives the bound

$$F_1^+(\theta)\ll H\sum_{M<p\le 2M}\sum_{1\le h\le H}\sum_{0\le h_2\le 2P}\left|\sum_{1\le z\le 2P-h_2} e(2^{-5}C\theta\Phi_1(z,h,h_2,p))\right|,$$

where

$$\Phi_1(z,h,h_2,p)=\Phi(z+h_2,h,p)-\Phi(z,h,p).$$

Thus, in a manner resembling the derivation of equation (2.37) of Vaughan [14], and imitating the argument leading to equation (3.1) and Lemmata 3.1 and 3.2 of Vaughan [14], we obtain

$$F_1^+(\theta)\ll MH^2P^{3/2}+M^{1/2}H^{3/2}P^{1/2}|F_3^+(\theta)|^{1/2}, \tag{2.14}$$

where $F_3^+(\theta)$ is bounded in the shape

$$|F_3^+(\theta)|^2 \le D(\theta)E(\theta) \tag{2.15}$$

for certain exponential sums $D(\theta)$ and $E(\theta)$. Here, it suffices for us to note that whenever $|\theta - r/q| \le q^{-2}$ and $(r,q)=1$, the exponential sum $D(\theta)$ satisfies

$$D(\theta) \ll P^{\varepsilon}\left(\frac{P^4H}{q+Q^5|q\theta - r|} + P^3H + q + Q^5|q\theta - r|\right), \tag{2.16}$$

and, whenever $M^5 \le X \le Q^5M^{-5}$, $(r,q)=1$, $q \le X$ and $|\theta - r/q| \le q^{-1}X^{-1}$, the exponential sum $E(\theta)$ satisfies

$$E(\theta) \ll P^{\varepsilon}\left(\frac{P^2HM^2}{(q+Q^5|q\theta - r|)^{1/5}} + P^2HM\right). \tag{2.17}$$

We remark here that the constant C in (2.13) is absorbed within the argument of the proofs of Lemmata 3.1 and 3.2 of Vaughan [14].

Suppose that $\theta \in \mathfrak{m}$. By Dirichlet's Theorem, we may choose $r \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(r,q)=1$, $q \le cP^{-1}Q^5$ and $|q\theta - r| \le c^{-1}PQ^{-5}$. By the definition of $\mathfrak{m}$, one has $q > c^{-1}P$, and so it follows from (2.1) and (2.15)–(2.17) that

$$|F_3^+(\theta)| \ll P^{\varepsilon}(P^3H + P^{-1}Q^5)^{1/2}(P^{9/5}HM^2 + P^2HM)^{1/2} \ll P^{5/2+\varepsilon}HM^{1/2},$$

whence by (2.14),

$$\sup_{\theta\in\mathfrak{m}} F_1^+(\theta) \ll P^{3/2}MH^2 + P^{7/4+\varepsilon}M^{3/4}H^2 \ll P^{7/4+\varepsilon}M^{3/4}H^2.$$

In view of (2.6), (2.9), and (2.13), we therefore deduce that

$$\sup_{\theta\in\mathfrak{m}} F_1^*(\theta) \ll P^2M + P^{7/4+\varepsilon}M^{3/4}H^2,$$

whence by (2.1) we obtain

$$\sup_{\theta\in\mathfrak{m}} F_1^*(\theta) \ll P^{2+\varepsilon}M. \tag{2.18}$$

The conclusion of the lemma is immediate on combining (2.12) and (2.18). □

We augment the previous estimates with a final mean value estimate that provides, in essence, a major arc bound.

Lemma 2.4. *Suppose that $u \ge 5/2$ and $C = A$ or B. Then one has*

$$\int_0^1 |\Delta_C(\theta)|^u\, d\theta \ll P^{\varepsilon}(P^2H^2M)^uQ^{-5}.$$

Proof. This estimate follows by applying the argument of the proof of Lemma 4.10 of Vaughan and Wooley [17]. One has merely to note that the function $\Delta_C(\theta)$ defined above carries all the variables save p with twice the weight appearing in the corresponding expression in the latter lemma, and that the constant C affects only the implicit constant in the claimed upper bound. □

The moment has arrived to unleash our forces on the proof of Theorem 2.1. We begin by extracting the efficient difference, and here we follow a routine originating in work of Vaughan [13] and applied in a situation similar to that at hand in Wooley [19]. Suppose initially that p is a fixed prime number with $M < p \le 2M$ and $p \equiv -1 \pmod 5$. Write

$$\mathcal{L}_t(p) = \iint\limits_{[0,1]^2} |F_p(A\alpha)F_p(B\beta)|^2 |f(\Lambda_1 p^5) f(\Lambda_2 p^5)|^{15-t} d\alpha\, d\beta.$$

By a change of variable, one finds that

$$\mathcal{L}_t(p) = p^{-10} \iint\limits_{[0,p^5]^2} |F_p(A\alpha p^{-5})F_p(B\beta p^{-5})|^2 |f(\Lambda_1) f(\Lambda_2)|^{15-t} d\alpha\, d\beta.$$

But $\Lambda_1(\alpha,\beta)$ and $\Lambda_2(\alpha,\beta)$ are linear forms in α and β with integral coefficients, so by the periodicity modulo 1 of $f(\theta)$ with respect to θ, one finds that

$$\begin{aligned}\mathcal{L}_t(p) &= p^{-10} \sum_{u=1}^{p^5} \sum_{v=1}^{p^5} \iint\limits_{[0,1]^2} \left| F_p\left(\frac{A(\alpha+u)}{p^5}\right) F_p\left(\frac{B(\beta+v)}{p^5}\right)\right|^2 |f(\Lambda_1) f(\Lambda_2)|^{15-t} d\alpha\, d\beta \\ &= \iint\limits_{[0,1]^2} \mathcal{G}_1(\alpha;p;A)\mathcal{G}_1(\beta;p;B) |f(\Lambda_1) f(\Lambda_2)|^{15-t} d\alpha\, d\beta, \qquad (2.19)\end{aligned}$$

where $\mathcal{G}_1(\theta;p;C)$ is the exponential sum defined in (2.5).

Observe next that on recalling (2.6), an application of Cauchy's inequality reveals that

$$\sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \mathcal{G}_1(\alpha;p;A)\mathcal{G}_1(\beta;p;B) \le (F_1^*(\alpha)F_1^*(\beta))^{1/2}.$$

By Lemma 2.3, therefore, we deduce that

$$\sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \mathcal{G}_1(\alpha;p;A)\mathcal{G}_1(\beta;p;B) \ll (P^{2+\varepsilon}M + \Delta_A(\alpha))^{1/2}(P^{2+\varepsilon}M + \Delta_B(\beta))^{1/2}.$$

Thus, on substituting into (2.19) and recalling (2.4), we conclude that

$$\iint\limits_{[0,1]^2} \mathcal{F}_t(\alpha,\beta)\, d\alpha\, d\beta \ll P^{2+\varepsilon}MI_1 + P^{1+\varepsilon}M^{1/2}(I_2+I_3) + I_4, \qquad (2.20)$$

where

$$I_1 = \iint\limits_{[0,1]^2} |f(\Lambda_1)f(\Lambda_2)|^{15-t}\, d\alpha\, d\beta, \qquad (2.21)$$

$$I_2 = \iint\limits_{[0,1]^2} \Delta_A(\alpha)^{1/2} |f(\Lambda_1)f(\Lambda_2)|^{15-t}\, d\alpha\, d\beta, \qquad (2.22)$$

$$I_3 = \iint\limits_{[0,1]^2} \Delta_B(\beta)^{1/2} |f(\Lambda_1)f(\Lambda_2)|^{15-t}\, d\alpha\, d\beta, \qquad (2.23)$$

and

$$I_4 = \iint_{[0,1]^2} \Delta_A(\alpha)^{1/2}\Delta_B(\beta)^{1/2}|f(\Lambda_1)f(\Lambda_2)|^{15-t}\,d\alpha\,d\beta. \tag{2.24}$$

The integral I_1 is easily disposed of by an application of Lemma 2.2. Since the linear forms Λ_1 and Λ_2 are linearly independent, we may combine a non-singular change of variables in (2.21) with Hölder's inequality to deduce that

$$I_1 \ll \iint_{[0,1]^2} |f(\xi)f(\zeta)|^{15-t}\,d\xi\,d\zeta \le \left(\int_0^1 |f(\alpha)|^{14}d\alpha\right)^{1+t}\left(\int_0^1 |f(\beta)|^{16}d\beta\right)^{1-t}$$
$$\ll \left(Q^{9.272729}\right)^{1+t}\left(Q^{11.077363}\right)^{1-t}.$$

Consequently, on recalling our assumption that $0 \le t \le 10^{-3}$, we find that

$$I_1 \ll Q^{20.3503-2t}. \tag{2.25}$$

We estimate the integral I_2 by applying Hölder's inequality once more. Thus, from (2.22) we obtain

$$I_2 \ll \Big(\sup_{(\alpha,\beta)\in[0,1]^2} |f(\Lambda_1)|\Big)^{3/5-t} I_5^{1/5} I_6^{(3-5t)/10} I_7^{(1+t)/2}, \tag{2.26}$$

where

$$I_5 = \iint_{[0,1]^2} \Delta_A(\alpha)^{5/2}|f(\Lambda_2)|^{16}\,d\alpha\,d\beta, \tag{2.27}$$

$$I_6 = \iint_{[0,1]^2} |f(\Lambda_1)^{18}f(\Lambda_2)^{16}|\,d\alpha\,d\beta \quad \text{and} \quad I_7 = \iint_{[0,1]^2} |f(\Lambda_1)^{18}f(\Lambda_2)^{14}|\,d\alpha\,d\beta.$$

By a change of variables, we again deduce from Lemma 2.2 that

$$I_6 \ll \left(\int_0^1 |f(\alpha)|^{18}d\alpha\right)\left(\int_0^1 |f(\beta)|^{16}d\beta\right) \ll Q^{24.077363} \tag{2.28}$$

and

$$I_7 \ll \left(\int_0^1 |f(\alpha)|^{18}d\alpha\right)\left(\int_0^1 |f(\beta)|^{14}d\beta\right) \ll Q^{22.272729}. \tag{2.29}$$

Also, by another change of variable, our assumption that $d \ne 0$ leads to the identity

$$\int_0^1 |f(\Lambda_2)|^{16}d\beta = \int_0^1 |f(d\beta)|^{16}d\beta = \int_0^1 |f(\beta)|^{16}d\beta.$$

Thus we deduce from (2.27) and Lemmata 2.2 and 2.4 that

$$I_5 \ll Q^{11.077363}\int_0^1 \Delta_A(\alpha)^{5/2}d\alpha \ll P^\varepsilon(P^2H^2M)^{5/2}Q^{6.077363}. \tag{2.30}$$

On substituting (2.28)–(2.30) into (2.26), and employing a trivial estimate for $f(\Lambda_1)$, we find that

$$I_2 \ll P^\varepsilon(P^2H^2M)^{1/2}Q^{20.17515-2t}. \tag{2.31}$$

Here again we make use of the assumption that $0 \le t \le 10^{-3}$.

Plainly, on interchanging the roles of α and β, and of A and B, in (2.23), the argument applied in the previous paragraph establishes in like manner that

$$I_3 \ll P^\varepsilon (P^2H^2M)^{1/2} Q^{20.17515-2t}. \tag{2.32}$$

Finally, applying Hölder's inequality yet again, we find from (2.24) that

$$I_4 \ll \Big(\sup_{(\alpha,\beta)\in[0,1]^2} |f(\Lambda_1)f(\Lambda_2)| \Big)^{3/5-t} I_8^{1/5} I_9^{4/5}, \tag{2.33}$$

where

$$I_8 = \iint\limits_{[0,1]^2} \big(\Delta_A(\alpha)\Delta_B(\beta)\big)^{5/2}\, d\alpha\, d\beta \quad \text{and} \quad I_9 = \iint\limits_{[0,1]^2} |f(\Lambda_1)f(\Lambda_2)|^{18}\, d\alpha\, d\beta.$$

By a change of variables, Lemma 2.2 on this occasion leads to the estimate

$$I_9 \ll \left(\int_0^1 |f(\alpha)|^{18} d\alpha \right) \left(\int_0^1 |f(\beta)|^{18} d\beta \right) \ll Q^{26}. \tag{2.34}$$

On the other hand, by Lemma 2.4, one has

$$I_8 \ll \big(P^\varepsilon (P^2H^2M)^{5/2} Q^{-5} \big)^2 \ll P^\varepsilon (P^2H^2M)^5 Q^{-10}. \tag{2.35}$$

Thus, on substituting (2.34) and (2.35) into (2.33), we arrive at the estimate

$$I_4 \ll P^{2+\varepsilon} H^2 M Q^{20-2t}. \tag{2.36}$$

We now combine the estimates (2.25), (2.31), (2.32), (2.36) and (2.20) to find that

$$\iint\limits_{[0,1]^2} \mathcal{F}_t(\alpha,\beta)\, d\alpha\, d\beta \ll P^{2+\varepsilon} M \left(Q^{20.3503-2t} + HQ^{20.17515-2t} + H^2 Q^{20-2t} \right).$$

On recalling (2.1), the desired conclusion

$$\iint\limits_{[0,1]^2} \mathcal{F}_t(\alpha,\beta)\, d\alpha\, d\beta \ll M P^{\varepsilon-6} Q^{30-2t}$$

follows with a modicum of computation.

Before departing this section, we record a further auxiliary estimate of use in the pruning argument described in §5. As an analogue of the exponential sum $\mathcal{F}_t(\alpha,\beta)$ defined in (2.4), we now write

$$\widehat{\mathcal{F}}_t(\alpha,\beta) = \sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} |F_p(A\alpha)|^2 |f(\Lambda_1 p^5) f(\Lambda_2 p^5)|^{15-t}. \tag{2.37}$$

Lemma 2.5. *Suppose that t is a real number with $0 \le t \le 10^{-3}$. Then whenever $\eta > 0$ is sufficiently small, one has the estimate*

$$\iint\limits_{[0,1]^2} \widehat{\mathcal{F}}_t(\alpha,\beta) d\alpha\, d\beta \ll PMQ^{20.3517-2t}.$$

Proof. We apply the argument underlying the proof of Theorem 2.1, making simple modifications as needed. Suppose first that p is a fixed prime number with $M < p \le 2M$ and $p \equiv -1 \pmod 5$, and write

$$\widehat{\mathcal{L}}_t(p) = \iint\limits_{[0,1]^2} |F_p(A\alpha)|^2 |f(\Lambda_1 p^5) f(\Lambda_2 p^5)|^{15-t} d\alpha\, d\beta.$$

As in the argument leading to (2.19), a change of variables leads to the identity

$$\widehat{\mathcal{L}}_t(p) = \iint\limits_{[0,1]^2} \mathcal{G}_1(\alpha; p; A) |f(\Lambda_1) f(\Lambda_2)|^{15-t} d\alpha\, d\beta. \tag{2.38}$$

But an application of Cauchy's inequality reveals that

$$\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \mathcal{G}_1(\alpha; p; A) \le M^{1/2} F_1^*(\alpha)^{1/2},$$

whence by Lemma 2.3 we obtain

$$\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \mathcal{G}_1(\alpha; p; A) \ll P^{1+\varepsilon} M + M^{1/2} \Delta_A(\alpha)^{1/2}.$$

On substituting into (2.38) and recalling (2.37), we deduce that

$$\iint\limits_{[0,1]^2} \widehat{\mathcal{F}}_t(\alpha, \beta) d\alpha\, d\beta \ll P^{1+\varepsilon} M I_1 + M^{1/2} I_2,$$

where I_1 and I_2 are defined, respectively, in (2.21) and (2.22). We therefore conclude from (2.25) and (2.31) that

$$\iint\limits_{[0,1]^2} \widehat{\mathcal{F}}_t(\alpha, \beta) d\alpha\, d\beta \ll P^{1+\varepsilon} M Q^{20.3503-2t} + P^{1+\varepsilon} H M Q^{20.17515-2t}.$$

The conclusion of the lemma now follows from (2.1) with a smidgen of computation. □

3. Preliminaries to an Application of the Circle Method

Before applying the circle method to prove Theorem 1, we need to eliminate some relatively simple cases. First of all, we may clearly suppose in (1.1) that for each i at least one of c_i or d_i is non-zero. Further, we may assume that $s = 34$, since any superfluous variables may be set to zero at the outset. Next we need some information about the number of distinct coefficient ratios c_i/d_i present in (1.1).

Lemma 3.1. *If there is an extended real number r such that $c_i/d_i = r$ for 16 or more values of i, then the system (1.1) has a non-trivial integral solution.*

Proof. If some ratio r is repeated 16 or more times, then we may assume by relabelling that $c_i/d_i = r$ for $1 \le i \le 16$. But then by taking a linear combination of the two equations, we find that the system (1.1) is equivalent to the new system

$$\begin{aligned} c_1x_1^5 + \cdots + c_{17}x_{17}^5 + \cdots + c_{34}x_{34}^5 &= 0, \\ D_{17}x_{17}^5 + \cdots + D_{34}x_{34}^5 &= 0, \end{aligned} \tag{3.1}$$

where $D_i = d_1c_i - c_1d_i$ for $17 \le i \le 34$. From Gray [11], one knows that for every prime p, any diagonal quintic equation in 16 or more variables necessarily possesses non-trivial p-adic solutions. It therefore follows from the fifth power technology of Vaughan and Wooley [17] that a single additive equation in 17 or more variables is non-trivially soluble over the integers. Thus we can find $y_{17}, \dots, y_{34} \in \mathbb{Z}$, not all zero, such that

$$D_{17}y_{17}^5 + \cdots + D_{34}y_{34}^5 = 0.$$

Moreover, if we let $C_{17} = c_{17}y_{17}^5 + \cdots + c_{34}y_{34}^5$, then we can find integers $z_1, \dots, z_{17}$, not all zero, satisfying

$$c_1z_1^5 + \cdots + c_{16}z_{16}^5 + C_{17}z_{17}^5 = 0,$$

and it is easy to see that $\mathbf{x} = (z_1, \dots, z_{16}, z_{17}y_{17}, \dots, z_{17}y_{34})$ is a non-trivial solution of (3.1). This completes the proof of the lemma. □

For each i, write $r_i = c_i/d_i$. It is immediate from Lemma 3.1 that there are at least three distinct values among $r_1, \dots, r_{34}$. We may therefore rearrange variables in such a way that $r_1 \ne r_2$ and $r_3 \ne r_4$. Since we then have $c_1d_2 \ne c_2d_1$, it follows by taking linear combinations of the two equations that (1.1) is equivalent to a system of the form

$$\begin{aligned} c_1x_1^5 + c_3x_3^5 + c_4x_4^5 + \cdots + c_{34}x_{34}^5 &= 0, \\ d_2x_2^5 + d_3x_3^5 + d_4x_4^5 + \cdots + d_{34}x_{34}^5 &= 0, \end{aligned}$$

where $c_1d_2 \ne 0$. We therefore assume from now on that $c_2 = d_1 = 0$. After replacing one or both of x_1 and x_2 by $-x_1$ and $-x_2$, if necessary, we may further assume that $c_1 > 0$ and $d_2 > 0$.

In our application of the circle method, we will be concerned with the linear forms

$$\gamma_i = c_i\alpha + d_i\beta \qquad (1 \le i \le 34).$$

Suppose that $1 \le i < j \le 4$ and $5 \le k \le 34$. It is clear that whenever $r_i \ne r_j$ we can write

$$\gamma_k = u_k\gamma_i + v_k\gamma_j \tag{3.2}$$

for some $u_k, v_k \in \mathbb{Q}$, but we would often prefer to be able to take $u_k, v_k \in \mathbb{Z}$. To this end, we write

$$D = \prod_{\substack{1 \le i < j \le 4 \\ r_i \ne r_j}} |c_id_j - c_jd_i|$$

and make the change of variables $x_k \to Dx_k$ $(5 \le k \le 34)$. Then we may replace the coefficients c_k and d_k by D^5c_k and D^5d_k, and this ensures that in (3.2), one may take $u_k, v_k \in \mathbb{Z}$.

Finally, we need to consider some local solubility issues. Since $r_1 \ne r_2$, the linear system

$$c_1z_1 + \cdots + c_{34}z_{34} = d_1z_1 + \cdots + d_{34}z_{34} = 0 \tag{3.3}$$

has a 32-dimensional space of real solutions. But for each i, the space of solutions with $z_i = 0$ has dimension 31, since there are at least three distinct values among $r_1, \ldots, r_{34}$. Hence there is a real solution $(z_1, \ldots, z_{34})$ to (3.3) with no z_i equal to zero, and a real solution $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{34})$ to (1.1) is now obtained by taking fifth roots. Moreover, by replacing x_i by $-x_i$ if necessary, and hence (c_i, d_i) by $(-c_i, -d_i)$, we may assume that $\eta_i > 0$ for each i, and by homogeneity we may further assume that $\eta_i < 1$ for each i. Notice that our earlier assumption that $c_1 > 0$ and $d_2 > 0$ may be preserved here by replacing $(\mathbf{c}, \mathbf{d})$ by $(\pm\mathbf{c}, \pm\mathbf{d})$, for a suitable choice of signs. Clearly, such an $\boldsymbol{\eta}$ provides a non-singular real solution to the system (1.1).

With regard to p-adic solubility, we know from work of Cook [8] that the system (1.1) has a non-trivial p-adic solution whenever $p \neq 11$. Since 11-adic solubility is imposed as a hypothesis in Theorem 1, we may henceforth assume that (1.1) has a non-trivial p-adic solution for each prime p. Moreover, the argument of Davenport and Lewis [9], pages 114–115, shows that the latter implies the existence of a non-singular p-adic solution for each prime p, provided that every form in the pencil of the two forms in (1.1) explicitly contains at least 16 variables. However, if this latter hypothesis fails to hold, then a simplified version of the argument used in the proof of Lemma 3.1 may be applied to produce a non-trivial rational solution to the system (1.1).

For future reference, we summarize the results of this section so far in the following lemma.

Lemma 3.2. *Suppose that the conclusion of Theorem* 1 *holds when all of the conditions* (i)-(iv) *below are satisfied. Then the theorem holds in general.*

(i) *One has $s = 34$, $c_2 = d_1 = 0$, $c_1, d_2 > 0$, and $c_3d_4 - c_4d_3 \neq 0$.*
(ii) *Each distinct ratio $r_i = c_i/d_i$ (in the extended real numbers) occurs for at most* 15 *different indices i.*
(iii) *For each k with $5 \leq k \leq 34$ and all i and j with $1 \leq i < j \leq 4$ and $r_i \neq r_j$, there exist integers u_k and v_k such that $\gamma_k = u_k\gamma_i + v_k\gamma_j$.*
(iv) *The system* (1.1) *has a non-singular p-adic solution for every prime p and a non-singular real solution $\boldsymbol{\eta}$ with $0 < \eta_i < 1$ for each i.*

Assuming from now on that conditions (i)–(iv) of Lemma 3.2 are satisfied, we are now ready to describe our strategy for proving Theorem 1. Recall the definitions of $F_p(\theta)$ and $f(\theta)$ from (2.2) and (2.3). When $\mathfrak{B}$ is any measurable subset of the unit square $[0, 1)^2$, define

$$\mathcal{N}(\mathfrak{B}) = \sum_{\substack{M<p\leq 2M \\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} F_p(\gamma_1) \cdots F_p(\gamma_4) f(p^5\gamma_5) \cdots f(p^5\gamma_{34})\, d\alpha\, d\beta. \tag{3.4}$$

Further, let $N(P)$ denote the number of solutions of the system

$$\begin{aligned} c_1x_1^5 + \cdots + c_4x_4^5 + p^5(c_5y_5^5 + \cdots + c_{34}y_{34}^5) &= 0, \\ d_1x_1^5 + \cdots + d_4x_4^5 + p^5(d_5y_5^5 + \cdots + d_{34}y_{34}^5) &= 0, \end{aligned} \tag{3.5}$$

with

$$1 \leq x_i \leq P \quad (1 \leq i \leq 4), \quad y_j \in \mathcal{A}(Q, R) \quad (5 \leq j \leq 34),$$

$$M < p \leq 2M, \quad (p, x_1x_2x_3x_4) = 1 \quad \text{and} \quad p \equiv -1 \pmod 5.$$

Notice that by orthogonality one has $N(P) = \mathcal{N}([0,1)^2)$. We aim to establish the expected lower bound $N(P) \gg MP^{-6}Q^{30}(\log P)^{-1}$ by an application of the Hardy-Littlewood method. Since every solution of (3.5) automatically satisfies (1.1), this conclusion suffices to prove Theorem 1. We complete our initial skirmishing by describing the Hardy-Littlewood dissection underlying our application of the circle method. Write $\delta = 1/100$, and define the major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q,a,b) = \{(\alpha,\beta) \in [0,1)^2 : |q\alpha - a| \le P^\delta Q^{-5} \text{ and } |q\beta - b| \le P^\delta Q^{-5}\}, \tag{3.6}$$

with $0 \le a, b \le q \le P^\delta M^5$ and $(q,a,b) = 1$. It is clear from (2.1) that these intervals are pairwise disjoint. Further, write $\mathfrak{m} = [0,1)^2 \backslash \mathfrak{M}$ for the minor arcs. We remark, as will become apparent in due course, that while this set-up allows the minor arcs to be handled rather easily, the treatment of the major arcs entails a non-trivial pruning process.

4. The Minor Arcs

The estimation of the minor arc contribution $\mathcal{N}(\mathfrak{m})$ is easily accomplished with the aid of Theorem 2.1, and so we sally towards the desired estimate

$$\mathcal{N}(\mathfrak{m}) \ll MQ^{30}P^{-6-\nu}, \tag{4.1}$$

for some positive number ν, without further comment. First of all, it is easy to deduce from condition (ii) of Lemma 3.2 that there is a partition $\mathcal{P}$ of the set $\{5,6,7,\ldots,34\}$ into 15 two-element blocks, with the property that $\{k,\ell\} \in \mathcal{P} \Rightarrow r_k \ne r_\ell$. Hence by using the trivial inequality

$$|z_1 \cdots z_n| \le |z_1|^n + \cdots + |z_n|^n,$$

one finds that

$$|f(p^5\gamma_5)\cdots f(p^5\gamma_{34})| \le \sum_{\substack{5\le k<\ell\le 34\\ r_k\ne r_\ell}} |f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15}.$$

It therefore follows from (3.4) that for some k and ℓ with $r_k \ne r_\ell$ one has

$$\mathcal{N}(\mathfrak{m}) \ll \sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{m}} |F_p(\gamma_1)\cdots F_p(\gamma_4)||f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15}\,d\alpha\,d\beta,$$

and, by interchanging the roles of k and ℓ if necessary, we may suppose that $c_k d_\ell \ne 0$. When $\mathfrak{B} \subseteq [0,1)^2$ and t is a real number with $0 \le t \le 1$, write

$$\mathcal{N}_{0,t}(\mathfrak{B}) = \sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |F_p(\gamma_1)F_p(\gamma_2)|^2|f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15-t}\,d\alpha\,d\beta \tag{4.2}$$

and

$$\mathcal{N}_{1,t}(\mathfrak{B}) = \sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |F_p(\gamma_3)F_p(\gamma_4)|^2|f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15-t}\,d\alpha\,d\beta. \tag{4.3}$$

Then after two applications of the Cauchy-Schwarz inequalities, we find that

$$\mathcal{N}(\mathfrak{m}) \ll \mathcal{N}_{0,0}(\mathfrak{m})^{1/2}\mathcal{N}_{1,0}([0,1)^2)^{1/2}. \tag{4.4}$$

Now by condition (iii) of Lemma 3.2 we can write

$$\gamma_k = u_k\gamma_3 + v_k\gamma_4 \quad \text{and} \quad \gamma_\ell = u_\ell\gamma_3 + v_\ell\gamma_4$$

for some integers u_k, v_k, u_ℓ and v_ℓ. Moreover, a simple calculation shows that

$$u_kv_\ell - v_ku_\ell = \frac{c_kd_\ell - d_kc_\ell}{c_3d_4 - d_3c_4} \neq 0,$$

so that on making the change of variables $(\alpha,\beta) \to (\gamma_3,\gamma_4)$ in (4.3), we find that Theorem 2.1 applies (with $t = 0$) to show that

$$\mathcal{N}_{1,0}([0,1)^2) \ll MQ^{30}P^{\varepsilon-6}. \tag{4.5}$$

It therefore suffices to bound $\mathcal{N}_{0,0}(\mathfrak{m})$, and for this we require an estimate of Weyl-type. Although bounds of somewhat higher quality may be obtained by working harder, the following estimate is adequate for the purpose at hand.

Lemma 4.1. *For every integer p with $M < p \leq 2M$, one has*

$$\sup_{(\alpha,\beta)\in\mathfrak{m}} |f(p^5\gamma_k)f(p^5\gamma_\ell)| \ll Q^{2-\sigma},$$

where $\sigma = 3 \times 10^{-5}$.

Proof. Suppose that $(\alpha,\beta) \in \mathfrak{m}$. By Dirichlet's Theorem, there exist integers a_k, a_ℓ, q_k and q_ℓ satisfying $(q_k,a_k) = (q_\ell,a_\ell) = 1$,

$$1 \leq q_k, q_\ell \leq Q^5P^{-\delta/2}, \quad |p^5\gamma_kq_k - a_k| \leq P^{\delta/2}Q^{-5} \quad \text{and} \quad |p^5\gamma_\ell q_\ell - a_\ell| \leq P^{\delta/2}Q^{-5}.$$

On applying Lemma 3.1 of Wooley [20], and making use of Lemma 2.2 above, one obtains the estimate

$$|f(p^5\gamma_i)| \ll Q^{1+\varepsilon}(q_i^{-1} + Q^{-5/2} + q_iQ^{-5})^{1/162} \qquad (i = k,\ell). \tag{4.6}$$

Write

$$\Delta = c_kd_\ell - c_\ell d_k \quad \text{and} \quad C = 8(|c_k| + |c_\ell|)(|d_k| + |d_\ell|)|\Delta|.$$

If $q_k > C^{-1}P^{\delta/2}$ or $q_\ell > C^{-1}P^{\delta/2}$, then the lemma follows at once from (4.6). Suppose, on the other hand, that $q_k \leq C^{-1}P^{\delta/2}$ and $q_\ell \leq C^{-1}P^{\delta/2}$, and write $q = |\Delta|q_kq_\ell p^5$. Then since

$$\alpha = \Delta^{-1}(\gamma_kd_\ell - \gamma_\ell d_k) \quad \text{and} \quad \beta = \Delta^{-1}(c_k\gamma_\ell - c_\ell\gamma_k),$$

we find that

$$\|q\alpha\| = \|q_kq_\ell p^5(\gamma_kd_\ell - \gamma_\ell d_k)\| \leq |d_\ell|q_\ell\|q_kp^5\gamma_k\| + |d_k|q_k\|q_\ell p^5\gamma_\ell\| \leq P^\delta Q^{-5},$$

and similarly for $\|q\beta\|$. Thus, on noting that $q \leq P^\delta M^5$, we obtain a contradiction to our assumption that $(\alpha,\beta) \in \mathfrak{m}$, and this completes the proof of the lemma. □

We can now complete our analysis of the minor arcs. Let t and σ be positive numbers with $t \leq 10^{-3}$ and $\sigma \leq 3 \times 10^{-5}$. By applying Theorem 2.1 and Lemma 4.1, one finds that

$$\begin{aligned}\mathcal{N}_{0,0}(\mathfrak{m}) &\ll \Big(\max_{M<p\le 2M}\sup_{(\alpha,\beta)\in\mathfrak{m}}|f(p^5\gamma_k)f(p^5\gamma_\ell)|\Big)^t\\ &\quad\times\iint\limits_{[0,1]^2}\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}}|F_p(\gamma_1)F_p(\gamma_2)|^2|f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15-t}d\alpha\, d\beta\\ &\ll Q^{t(2-\sigma)}MQ^{30-2t}P^{\varepsilon-6}.\end{aligned}$$

Thus we obtain

$$\mathcal{N}_{0,0}(\mathfrak{m}) \ll MQ^{30}P^{-6-\tau}, \tag{4.7}$$

for some positive number τ, and on recalling (4.4) and (4.5), one arrives at the desired conclusion (4.1).

5. Pruning the Major Arcs

Although we have precise knowledge concerning the asymptotic behavior of the exponential sums $F_p(\gamma_i)$ throughout the set of major arcs defined by (3.6), such detailed information for the sums $f(p^5\gamma_i)$ is currently available only on a much thinner set. We must therefore perform a substantial amount of pruning. Specifically, we let $L=(\log P)^\delta$, and define $\mathfrak{N}$ to be the union of the intervals

$$\mathfrak{N}(q,a,b)=\{(\alpha,\beta)\in[0,1)^2 : |\alpha-a/q|\le LP^{-5} \text{ and } |\beta-b/q|\le LP^{-5}\}, \tag{5.1}$$

with $0\le a,b\le q\le L$ and $(q,a,b)=1$. Also, we take $\mathfrak{n}=[0,1)^2\backslash\mathfrak{N}$. We aim to show that

$$\mathcal{N}(\mathfrak{M}\backslash\mathfrak{N}) \ll MP^{-6}Q^{30}(\log P)^{-1-\tau} \tag{5.2}$$

for some positive number τ. In order to establish this bound, we first apply Cauchy's inequality as in the argument at the beginning of §4, thereby obtaining

$$\mathcal{N}(\mathfrak{M}\backslash\mathfrak{N}) \ll \mathcal{N}_{0,0}(\mathfrak{M}\backslash\mathfrak{N})^{1/2}\mathcal{N}_{1,0}([0,1)^2)^{1/2},$$

where $\mathcal{N}_{0,t}$ and $\mathcal{N}_{1,t}$ are defined as in (4.2) and (4.3), and where the indices k and ℓ occurring in those definitions satisfy $r_k\ne r_\ell$. But by making the change of variables $(\alpha,\beta)\to(\gamma_3,\gamma_4)$ in (4.3), just as in the argument leading to (4.5), we find that $\mathcal{N}_{1,0}([0,1)^2)$ is transformed into a mean value of the shape (4.2), and moreover the coefficients of the generating functions in this new mean value satisfy the same hypotheses as those imposed on the corresponding coefficients in (4.2). With a modicum of contemplation, therefore, one finds that whenever one can establish the estimates

$$\mathcal{N}_{0,0}([0,1)^2) \ll MP^{-6}Q^{30}(\log P)^{-1} \quad\text{and}\quad \mathcal{N}_{0,0}(\mathfrak{M}\backslash\mathfrak{N}) \ll MP^{-6}Q^{30}(\log P)^{-1-\tau}, \tag{5.3}$$

for some positive number τ, with sufficient uniformity in the underlying coefficients, then the estimate

$$\mathcal{N}_{1,0}([0,1)^2) \ll MP^{-6}Q^{30}(\log P)^{-1},$$

and hence also (5.2), will follow immediately. In the remainder of this section we establish the desired estimates (5.3) with the claimed uniformity, and hence achieve the hard pruning required to complete the analysis of the major arcs.

We begin by considering the major arc approximations to the functions $F_p(\gamma_i)$, and this requires some notation. When $1 \le i \le 34$, write

$$S_i(q,a,b) = \sum_{r=1}^{q} e((c_i a + d_i b) r^5/q), \tag{5.4}$$

$$S_i(q,a,b;p) = S_i(q,a,b) - p^{-1} S_i(q, ap^5, bp^5), \tag{5.5}$$

and

$$v_i(\xi,\zeta;B) = \int_0^B e((c_i\xi + d_i\zeta)\gamma^5)\, d\gamma. \tag{5.6}$$

We define the function $\Xi_i(p) = \Xi_i(\alpha,\beta;p)$ for $(\alpha,\beta) \in [0,1)^2$ by taking

$$\Xi_i(\alpha,\beta;p) = q^{-1} S_i(q,a,b;p) v_i(\alpha - a/q, \beta - b/q; P),$$

when $(\alpha,\beta) \in \mathfrak{M}(q,a,b) \subseteq \mathfrak{M}$, and otherwise by taking $\Xi_i(\alpha,\beta;p) = 0$.

Lemma 5.1. *When p is an integer with $M < p \le 2M$ and $(\alpha,\beta) \in \mathfrak{M}(q,a,b) \subseteq \mathfrak{M}$, one has*

$$|F_p(\gamma_i) - \Xi_i(\alpha,\beta;p)| \ll q^\varepsilon (q + P^5|q\alpha - a| + P^5|q\beta - b|)^{1/2}.$$

Proof. When $(\alpha,\beta) \in \mathfrak{M}(q,a,b) \subseteq \mathfrak{M}$ and $M < p \le 2M$, it follows from Theorem 4.1 of Vaughan [15] that

$$\begin{aligned} F_p(\gamma_i) &= \sum_{1\le x\le P} e((c_i\alpha + d_i\beta)x^5) - \sum_{1\le y \le P/p} e(p^5(c_i\alpha + d_i\beta)y^5) \\ &= q^{-1} S_i(q,a,b) v_i(\alpha - a/q, \beta - b/q; P) \\ &\quad - q^{-1} S_i(q, ap^5, bp^5) v_i(p^5(\alpha - a/q), p^5(\beta - b/q); P/p) \\ &\quad + O(q^\varepsilon (q + P^5|q\alpha - a| + P^5|q\beta - b|)^{1/2}). \end{aligned}$$

But a change of variables demonstrates that

$$v_i(p^5\xi, p^5\zeta; B) = p^{-1} v_i(\xi,\zeta;pB), \tag{5.7}$$

and the lemma now follows on recalling (5.5). □

As our first step in the pruning procedure, we replace the exponential sums $F_p(\gamma_i)$ by their major arc approximations $\Xi_i(\alpha,\beta;p)$. In this context, when t is a real number with $0 \le t \le 1$ and $\mathfrak{B} \subseteq [0,1)^2$, we write

$$\mathcal{N}^*_{0,t}(\mathfrak{B}) = \sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |\Xi_1(\alpha,\beta;p)\Xi_2(\alpha,\beta;p)|^2 |f(p^5\gamma_k) f(p^5\gamma_\ell)|^{15-t} d\alpha\, d\beta.$$

Lemma 5.2. *Suppose that t is a real number with $0 \le t \le 10^{-3}$. Then whenever $\mathfrak{B} \subseteq \mathfrak{M}$, one has*

$$\mathcal{N}_{0,t}(\mathfrak{B}) = \mathcal{N}^*_{0,t}(\mathfrak{B}) + O(MP^{-6-\tau} Q^{30-2t}),$$

for some positive number τ.

Proof. Suppose that $(\alpha,\beta)\in\mathfrak{M}(q,a,b)\subseteq\mathfrak{M}$. Then by Lemma 5.1, we have the estimate

$$|F_p(\gamma_i)-\Xi_i(\alpha,\beta;p)|\ll P^\delta M^{5/2}\quad (i=1,2).$$

On substituting into (4.2), we find that

$$|\mathcal{N}_{0,t}(\mathfrak{B})-\mathcal{N}^*_{0,t}(\mathfrak{B})|\ll P^{4\delta}M^{10}\mathcal{T}_0+P^{2\delta}M^5(\mathcal{T}_1+\mathcal{T}_2),$$

where

$$\mathcal{T}_0=\sum_{M<p\le 2M}\iint_{[0,1]^2}|f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15-t}d\alpha\,d\beta,$$

and for $i=1,2$, we write

$$\mathcal{T}_i=\sum_{\substack{M<p\le 2M\\ p\equiv -1\ (\mathrm{mod}\ 5)}}\iint_{[0,1]^2}|F_p(\gamma_i)|^2|f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15-t}d\alpha\,d\beta.$$

A change of variables reveals that

$$\mathcal{T}_0\ll M\iint_{[0,1]^2}|f(\xi)f(\zeta)|^{15-t}d\xi\,d\zeta.$$

Thus, as in the argument leading to (2.25), we find that

$$\mathcal{T}_0\ll MQ^{20.3503-2t}.$$

On the other hand, since $c_2=d_1=0$, it is apparent that $\mathcal{T}_i$ $(i=1,2)$ is a mean value of the type estimated in Lemma 2.5, whence

$$\mathcal{T}_i\ll PMQ^{20.3517-2t}\quad (i=1,2).$$

We therefore conclude that

$$|\mathcal{N}_{0,t}(\mathfrak{B})-\mathcal{N}^*_{0,t}(\mathfrak{B})|\ll P^{4\delta}M^{11}Q^{20.3503-2t}+P^{1+2\delta}M^6Q^{20.3517-2t},$$

and the desired conclusion follows from (2.1) with a smattering of computation. □

Lemma 5.3. *Suppose that t is a real number with $0\le t\le 10^{-3}$. Then for each positive number ε, one has*

$$\mathcal{N}^*_{0,t}([0,1)^2)\ll MP^{\varepsilon-6}Q^{30-2t}.$$

Proof. Under the hypotheses of the lemma, Theorem 2.1 shows that $\mathcal{N}_{0,t}([0,1)^2)\ll MP^{\varepsilon-6}Q^{30-2t}$. The desired conclusion is therefore immediate from Lemma 5.2. □

It transpires that a non-trivial analysis is required to establish (5.3). In particular, we need estimates for $f(p^5\gamma_i)$ that are valid over a somewhat larger range than has previously been dealt with in the literature. Fortunately, such estimates are obtainable by a simple modification of the argument of Lemma 7.2 of Vaughan and Wooley [16]. For ease of comparison with [16], we temporarily adopt the notation

$$g(\alpha)=\sum_{x\in\mathcal{A}(P,R)}e(\alpha x^k),$$

and write $\mathcal{L}=\log P$ and $\mathcal{L}_2=\log\log P$. The following provides the required extension of the aforementioned lemma.

Lemma 5.4. *Suppose that $2 \le R \le \mathcal{M} \le P$, and suppose also that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy $(a,q)=1$ and $q + P^k|q\alpha - a| \le T\mathcal{M}$. Then one has*

$$g(\alpha) \ll \mathcal{L}^3 q^\varepsilon \left(P(q+P^k|q\alpha-a|)^{-1/2k} + (P\mathcal{M}R)^{1/2} + PR^{1/2}(T/\mathcal{M})^{1/4} \right).$$

Proof. We apply the argument of the proof of Lemma 7.2 of [16], noting that in view of the first part of Theorem 4.1 of Vaughan [15] the estimate (7.5) of [16] may be replaced by the upper bound

$$S_2 \ll S_3 + \mathcal{L}_2 UV + RU^2 q^\varepsilon \left(q + (4UV)^k |q\alpha - a| \right)^{1/2}.$$

Moreover, one has $U < P/V$, so it follows from the hypothesis in the statement of the lemma that

$$S_2 \ll S_3 + \mathcal{L}_2 UV + RU^2 q^\varepsilon (T\mathcal{M})^{1/2}. \tag{5.8}$$

It now suffices to note that the upper bound provided in (5.8) differs from that in (7.5) of [16] only insofar as the third term of (5.8) is replaced by $RU^2 q^{1/2+\varepsilon}$ in (7.5) of [16]. Since $\mathcal{M} \le V < \mathcal{M}R$, one finds from (7.1) and (7.3) of [16] that

$$g(\alpha) \ll \mathcal{L}^3 q^\varepsilon \left(P(q+P^k|q\alpha-a|)^{-1/2k} + (P\mathcal{M}R)^{1/2} \right) + \Sigma,$$

where the new term Σ is bounded in the shape

$$\Sigma \ll \mathcal{L}^2 (V\mathcal{L}_2 RU^2 q^\varepsilon (T\mathcal{M})^{1/2})^{1/2} \ll \mathcal{L}^3 q^\varepsilon PR^{1/2}(T/\mathcal{M})^{1/4},$$

and the claimed version of the lemma now follows. □

We are now in a position to obtain estimates for $f(p^5\gamma_i)$ when $(\alpha,\beta) \in \mathfrak{M}$.

Lemma 5.5. *Suppose that $(\alpha,\beta) \in \mathfrak{M}(q,a,b) \subseteq \mathfrak{M}$ and that p is an integer with $M < p \le 2M$. Write $\Lambda_i = c_i a + d_i b$, and suppose in addition that η is a sufficiently small positive number, and that σ is a positive number with $\sigma \le 1/10$. Then the estimate*

$$f(p^5\gamma_i) \ll \frac{Q(q,p^5\Lambda_i)^{1/10-\sigma}}{(q+P^5|q\gamma_i - \Lambda_i|)^{1/10-\sigma}} + Q^{1-\delta/5}$$

holds uniformly for $q \le P^\delta M^5$.

Proof. Write $A = 4\sigma^{-1}$ and $D = (q, p^5\Lambda_i)$, and suppose first of all that

$$q + P^5|q\gamma_i - \Lambda_i| \ge D(\log Q)^A. \tag{5.9}$$

We seek to apply Lemma 5.4 with $\mathcal{M} = P^\delta M^{5/2}$ and $T = CM^{5/2}$, for some suitable positive constant C. On writing $q' = q/D$ and $\Lambda_i' = p^5\Lambda_i/D$, we find that

$$q' + Q^5|q'p^5\gamma_i - \Lambda_i'| \le P^\delta M^5 + 32P^5(|c_i(q\alpha - a)| + |d_i(q\beta - b)|),$$

so that on taking $C = 1 + 64(|c_i| + |d_i|)$, it follows from (3.6) that Lemma 5.4 applies with $\mathcal{M}$ and T as above. We therefore deduce that

$$f(p^5\gamma_i) \ll (\log Q)^3 (q')^\varepsilon \left(\frac{Q}{(q'+Q^5|q'p^5\gamma_i - \Lambda_i'|)^{1/10}} + R^{1/2}\left(P^{\delta/2}Q^{1/2}M^{5/4} + Q^{1-\delta/4} \right) \right),$$

and the lemma now follows under the assumption (5.9). Now suppose instead that (5.9) does not hold. Then by Lemma 8.5 of Vaughan and Wooley [16], one has

$$f(p^5\gamma_i) \ll \frac{(q')^\varepsilon Q}{(q' + Q^5|q'p^5\gamma_i - \Lambda_i'|)^{1/5}} + Q\exp(-c\sqrt{\log Q})(1 + Q^5|p^5(\gamma_i - \Lambda_i/q)|),$$

where the constant c may depend on η and A. Since (5.9) fails to hold, one sees easily that the first term in the above expression dominates the second, and the lemma now follows easily in this case as well. □

We are now ready to replace the functions $f(p^5\gamma_i)$ by the approximations given in Lemma 5.5. For convenience, we take $\sigma = 1/110$ in our application of Lemma 5.5, this being sufficient for our purposes. We also introduce the function $\Delta_p(\gamma_i)$, which we define for $(\alpha,\beta) \in [0,1)^2$ by taking

$$\Delta_p(\gamma_i) = \frac{Q(q, p^5\Lambda_i)^{1/11}}{(q + P^5|q\gamma_i - \Lambda_i|)^{1/11}},$$

when $(\alpha,\beta) \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, and otherwise by taking $\Delta_p(\gamma_i) = 0$. Here, as in the statement of Lemma 5.5, we write $\Lambda_i = c_i a + d_i b$. The following lemma allows us to replace $f(p^5\gamma_i)$ by $\Delta_p(\gamma_i)$.

Lemma 5.6. *Whenever $\mathfrak{B} \subseteq \mathfrak{M}$, one has*

$$\mathcal{N}_{0,0}(\mathfrak{B}) \ll \sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |\Xi_1(p)\Xi_2(p)|^2(\Delta_p(\gamma_k)\Delta_p(\gamma_\ell))^{15} d\alpha\, d\beta + MQ^{30}P^{-6-\tau},$$

for some positive number τ.

Proof. Suppose that $\mathfrak{B} \subseteq \mathfrak{M}$, and let t be a positive number with $t \le 10^{-3}$. By Lemma 5.5, one has for each prime p with $M < p \le 2M$,

$$|f(p^5\gamma_k)f(p^5\gamma_\ell)|^t \ll (\Delta_p(\gamma_k)^t + Q^{t(1-\delta/5)})(\Delta_p(\gamma_\ell)^t + Q^{t(1-\delta/5)}),$$

whence it follows from the trivial estimate $\Delta_p(\gamma_i) \ll Q$ that

$$\mathcal{N}^*_{0,0}(\mathfrak{B}) \ll \sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |\Xi_1(p)\Xi_2(p)|^2|f(p^5\gamma_k)f(p^5\gamma_\ell)|^{15-t}(\Delta_p(\gamma_k)\Delta_p(\gamma_\ell))^t d\alpha\, d\beta + Q^{t(2-\delta/5)}\mathcal{N}^*_{0,t}([0,1)^2).$$

If the second term on the right hand side of this inequality dominates the first, then the proof of the lemma follows immediately from Lemma 5.3. Otherwise, following two applications of Hölder's inequality, one finds that

$$\mathcal{N}^*_{0,0}(\mathfrak{B}) \ll \mathcal{N}^*_{0,0}(\mathfrak{B})^{1-t/15}\Bigg(\sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |\Xi_1(p)\Xi_2(p)|^2(\Delta_p(\gamma_k)\Delta_p(\gamma_\ell))^{15} d\alpha\, d\beta\Bigg)^{t/15},$$

whence

$$\mathcal{N}^*_{0,0}(\mathfrak{B}) \ll \sum_{\substack{M<p\le 2M \\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{B}} |\Xi_1(p)\Xi_2(p)|^2(\Delta_p(\gamma_k)\Delta_p(\gamma_\ell))^{15} d\alpha\, d\beta.$$

The conclusion of the lemma is now immediate from Lemma 5.2. □

Before concluding our pruning operation, we pause to evaluate a sum and an integral that are critical to the remainder of our analysis. Define

$$S(q)=\sum_{a=1}^{q}\sum_{\substack{b=1\\(q,a,b)=1}}^{q} q^{-30/11}(q,c_ka+d_kb)^{15/11}(q,c_\ell a+d_\ell b)^{15/11}. \tag{5.10}$$

Also, when $\mathfrak{W}$ denotes either $\mathfrak{M}$ or $\mathfrak{M}\backslash\mathfrak{N}$, define

$$I(q,a,b;\mathfrak{W})=\iint_{\mathfrak{W}(q,a,b)}(1+P^5|\gamma_k-\Lambda_k/q|)^{-15/11}(1+P^5|\gamma_\ell-\Lambda_\ell/q|)^{-15/11}d\alpha\,d\beta, \tag{5.11}$$

where $\mathfrak{W}(q,a,b)$ denotes $\mathfrak{M}(q,a,b)$ when $\mathfrak{W}=\mathfrak{M}$, and $\mathfrak{W}(q,a,b)$ denotes

$$\mathfrak{M}(q,a,b)\backslash\mathfrak{N}(q,a,b)$$

when $\mathfrak{W}=\mathfrak{M}\backslash\mathfrak{N}$. Here again, we write Λ_i for c_ia+d_ib $(i=k,\ell)$.

Lemma 5.7. *The function $S(q)$ is multiplicative, and one has*

$$S(q)\ll q^{-1/3}.$$

Proof. Suppose that q and r are natural numbers with $(q,r)=1$. Following the pattern provided in Lemmata 2.10 and 2.11 of Vaughan [15], given integers a and b, we may apply Euclid's algorithm to obtain unique integers u, v, x, y, with $1\le u,x\le r$ and $1\le v,y\le q$, such that

$$a\equiv uq+vr \pmod{qr}\qquad\text{and}\qquad b\equiv xq+yr \pmod{qr}.$$

Furthermore, one has $(qr,a,b)=1$ if and only if $(q,v,y)=1$ and $(r,u,x)=1$. On changing variables in (5.10) and noting that $(qr,Cq+Dr)=(q,D)(r,C)$, it now follows easily that $S(qr)=S(q)S(r)$. This establishes that the function $S(q)$ is multiplicative.

Suppose next that π is a prime number, and let h be a positive integer. Consider the sum (5.10) with $q=\pi^h$, and consider a fixed choice of a and b. Suppose that $\pi^A\|(c_ka+d_kb)$ and $\pi^B\|(c_\ell a+d_\ell b)$. We may assume without loss of generality that $A\le B$, and further, in view of the condition $(\pi^h,a,b)=1$ imposed in the sum (5.10), that π does not divide b. On eliminating a between the congruences

$$c_ka+d_kb\equiv 0 \pmod{\pi^A}\quad\text{and}\quad c_\ell a+d_\ell b\equiv 0 \pmod{\pi^B},$$

we therefore deduce that $c_kd_\ell-d_kc_\ell\equiv 0 \pmod{\pi^A}$. But $r_k\ne r_\ell$, and so the left hand side of this congruence is non-zero. Consequently, one finds that π^A is absolutely bounded in terms of the coefficients c_i and d_i $(i=k,\ell)$. Next we note that $c_\ell a\equiv -d_\ell b \pmod{\pi^B}$. Since $d_\ell\ne 0$, one finds that (π^B,d_ℓ) is absolutely bounded in terms of d_ℓ. But after dividing through by (π^B,d_ℓ) and fixing a, one sees that the residue class of b modulo $\pi^B/(\pi^B,d_\ell)$ is determined. We therefore conclude from this discussion that when $\pi^A\|(c_ka+d_kb)$ and $\pi^B\|(c_\ell a+d_\ell b)$, then there is no loss of generality in supposing that $\pi^A\ll 1$ and that the total number of possible choices for a and b is $O(\pi^{2h-B})$. We may thus infer that

$$S(\pi^h)\ll\sum_{A\le B\le h}\pi^{2h-B}\pi^{-30h/11}(\pi^{A+B})^{15/11}\ll h^2\pi^{-4h/11}.$$

The final assertion of the lemma now follows from the multiplicativity of $S(q)$ already established, since this yields $S(q) \ll q^{\varepsilon-4/11} \ll q^{-1/3}$. $\square$

Lemma 5.8. *Suppose that $\mathfrak{W}$ is either $\mathfrak{M}$ or $\mathfrak{M}\backslash\mathfrak{N}$, and define Y by taking*

$$Y = \begin{cases} 1, & \text{when } \mathfrak{W} = \mathfrak{M}, \text{ or when } \mathfrak{W} = \mathfrak{M}\backslash\mathfrak{N} \text{ and } q > L, \\ L, & \text{when } \mathfrak{W} = \mathfrak{M}\backslash\mathfrak{N} \text{ and } q \le L. \end{cases}$$

Then one has

$$I(q, a, b; \mathfrak{W}) \ll P^{-10} Y^{-4/11}.$$

Proof. Suppose that $\mathfrak{W}$ is either $\mathfrak{M}$ or $\mathfrak{M}\backslash\mathfrak{N}$, and define Y as in the statement of the lemma. Then on setting $W = 0$ when $\mathfrak{W} = \mathfrak{M}$, or when $\mathfrak{W} = \mathfrak{M}\backslash\mathfrak{N}$ and $q > L$, and otherwise setting $W = L$, it follows by making a change of variables that

$$I(q, a, b; \mathfrak{W}) \ll \underset{\max\{|\xi|,|\zeta|\} \ge WP^{-5}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} (1 + P^5|c_k\xi + d_k\zeta|)^{-15/11}(1 + P^5|c_\ell\xi + d_\ell\zeta|)^{-15/11} d\xi\, d\zeta.$$

Since $c_k d_\ell - c_\ell d_k \neq 0$, a second change of variables reveals that

$$I(q, a, b; \mathfrak{W}) \ll P^{-10} \underset{\max\{\mu,\nu\} \ge \lambda W}{\int_0^{\infty}\int_0^{\infty}} (1 + \mu)^{-15/11}(1 + \nu)^{-15/11} d\mu\, d\nu,$$

where λ is a positive number depending at most on c_i and d_i $(i = k, \ell)$. Thus we conclude that

$$I(q, a, b; \mathfrak{W}) \ll P^{-10} \int_{\lambda W}^{\infty} (1 + \omega)^{-15/11} d\omega \ll P^{-10} Y^{-4/11},$$

as desired. $\square$

As a final preparation for the impending pruning operation, we sharpen the information concerning the function $\Xi_1(p)\Xi_2(p)$ available to us on the major arcs $\mathfrak{M}$, paying particular attention to those $\mathfrak{M}(q, a, b)$ with q divisible by p. Recall from Lemma 3.2 (i) that $c_2 = d_1 = 0$. It is therefore convenient to introduce the notation

$$S(q, a) = \sum_{r=1}^{q} e(ar^5/q) \tag{5.12}$$

and

$$S(q, a; p) = S(q, a) - p^{-1} S(q, ap^5).$$

Finally, we define the multiplicative function $\kappa(q)$ on prime powers π^h by taking

$$\kappa(\pi^h) = \begin{cases} 4\pi^{-1/2}, & \text{when } h = 1, \\ \pi^{-h/5}, & \text{when } h > 1. \end{cases}$$

Lemma 5.9. *Suppose that p is a prime number with $M < p \le 2M$ and $p \equiv -1 \pmod 5$. Write*

$$\Upsilon_p(q, a, b) = \begin{cases} \kappa(q), & \text{when } p \nmid q, \\ p^{-2}(ab, p)\kappa(p^{-1}q), & \text{when } p \| q, \\ 0, & \text{when } p^2 | q. \end{cases}$$

Then whenever $(\alpha,\beta)\in\mathfrak{M}(q,a,b)\subseteq\mathfrak{M}$*, one has*

$$|\Xi_1(\alpha,\beta;p)\Xi_2(\alpha,\beta;p)|\ll P^2\Upsilon_p(q,a,b).$$

Proof. Suppose that $(\alpha,\beta)\in\mathfrak{M}(q,a,b)\subseteq\mathfrak{M}$. Then by making a trivial estimate for $v_i(\xi,\zeta;P)$, one obtains from the definition of $\Xi_i(\alpha,\beta;p)$ the estimate

$$|\Xi_1(p)\Xi_2(p)|\ll P^2q^{-2}S(q,c_1a;p)S(q,d_2b;p). \tag{5.13}$$

Observe next that when $\mathfrak{M}(q,a,b)\subseteq\mathfrak{M}$, then one has $(q,a,b)=1$. Since $p>M$, we may therefore suppose without loss of generality that $(p,c_1a)=1$. Also, since $q\le P^\delta M^5$, one finds that $p^6\nmid q$. Define j by taking $p^j=(q,p^5)$, and write $q_j=qp^{-j}$. Then we deduce from Lemma 4.5 of Vaughan [15] that

$$S(q,c_1a)=S(q_j,c_1ap^{4j})S(p^j,c_1aq_j^4).$$

Moreover, by two changes of variable, one finds that

$$S(q,c_1ap^5)=p^jS(q_j,c_1ap^{5-j})=p^jS(q_j,c_1ap^{4j}).$$

Thus we deduce that

$$S(q,c_1a;p)=S(q_j,c_1ap^{4j})(S(p^j,c_1aq_j^4)-p^{j-1}). \tag{5.14}$$

When $j\ge 2$, it follows from Lemma 4.4 of [15] that $S(p^j,c_1aq_j^4)=p^{j-1}$, whence the relation (5.14) yields

$$q^{-2}S(q,c_1a;p)S(q,d_2b;p)=0. \tag{5.15}$$

Suppose next that $j=1$. Since $p\equiv -1 \pmod 5$, every element of $(\mathbb{Z}/p\mathbb{Z})^\times$ is a fifth-power residue, and thus it follows that $S(p,c_1aq_1^4)=0$ (or see Lemma 4.3 of [15]). We therefore deduce from (5.14) that

$$S(q,c_1a;p)=-S(q_1,c_1ap^4),$$

and with little additional effort one also deduces that

$$|S(q,d_2b;p)|\le (b,p)|S(q_1,d_2bp^4)|.$$

Thus we deduce from Lemma 4.3 and Theorem 4.2 of Vaughan [15] that

$$\begin{aligned}|q^{-2}S(q,c_1a;p)S(q,d_2b;p)|&\le |p^{-2}(b,p)q_1^{-2}S(q_1,c_1ap^4)S(q_1,d_2bp^4)|\\&\ll p^{-2}(b,p)\left(\frac{\kappa(q_1)}{\kappa((q_1,c_1ap^4))}\right)\left(\frac{\kappa(q_1)}{\kappa((q_1,d_2bp^4))}\right).\end{aligned}$$

But since $(q,a,b)=1$, one has $(q_1,c_1ap^4,d_2bp^4)\ll 1$, and hence

$$\kappa((q_1,c_1ap^4))\kappa((q_1,d_2bp^4))\gg \kappa(q_1).$$

We therefore conclude that in this case

$$|q^{-2}S(q,c_1a;p)S(q,d_2b;p)|\ll p^{-2}(b,p)\kappa(q_1). \tag{5.16}$$

Finally, when $j=0$, it follows by a change of variable that

$$S(q,c_1a;p)S(q,d_2b;p)=(1-1/p)^2S(q,c_1a)S(q,d_2b),$$

and we find, as in the treatment of the case $j=1$ above, that

$$|q^{-2}S(q,c_1a;p)S(q,d_2b;p)|\ll \kappa(q). \tag{5.17}$$

The conclusion of the lemma follows immediately on collecting together (5.15)–(5.17), and substituting into (5.13). $\square$

Our collection of estimates now assembled, the end of the pruning process lies within our grasp. We begin by estimating $\mathcal{N}_{0,0}(\mathfrak{M})$, noting that with $\mathfrak{W}=\mathfrak{M}$, it follows from Lemmata 5.6 and 5.9 that

$$\mathcal{N}_{0,0}(\mathfrak{W}) \ll Q^{30}P^4 \sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \sum_{\substack{q\le P^\delta M^5\\ p^2\nmid q}} \Theta_p(q) + MQ^{30}P^{-6-\tau}, \tag{5.18}$$

for some positive number τ, where

$$\Theta_p(q)=\sum_{\substack{a=1\\ (q,a,b)=1}}^{q}\sum_{b=1}^{q} \Upsilon_p(q,a,b)^2 I(q,a,b;\mathfrak{W})m_p(q,a,b),$$

and

$$m_p(q,a,b)=q^{-30/11}(q,p^5(c_ka+d_kb))^{15/11}(q,p^5(c_\ell a+d_\ell b))^{15/11}.$$

Suppose first that $p\| q$, and write $q_1=p^{-1}q$. Then by Lemma 5.8 and the definition of $\Upsilon_p(q,a,b)$, we have

$$\Theta_p(q)\ll p^{-4}\kappa(q_1)^2P^{-10}\sum_{\substack{a=1\\ (q,a,b)=1}}^{q}\sum_{b=1}^{q} (p,ab)^2 q_1^{-30/11}(q_1,c_ka+d_kb)^{15/11}(q_1,c_\ell a+d_\ell b)^{15/11}.$$

On making use of Lemma 5.7, one finds that the contribution to the double sum arising from those terms with $p|a$ or $p|b$ is at most $p^3S(q_1)\ll p^3q_1^{-1/3}$, while the corresponding contribution arising from those terms with $(p,ab)=1$ is at most $p^2S(q_1)\ll p^2q_1^{-1/3}$. Thus we deduce that

$$\Theta_p(q)\ll p^{-1}P^{-10}\kappa(q_1)^2q_1^{-1/3}.$$

It follows that the contribution to (5.18) arising from those terms with $p\| q$ is of order

$$\begin{aligned} Q^{30}P^{-6}M^{-1}\sum_{M<p\le 2M}\sum_{q_1=1}^{\infty}\kappa(q_1)^2q_1^{-1/3} &\ll P^{-6}Q^{30}\prod_{\pi\text{ prime}}(1+19\pi^{-4/3})\\ &\ll P^{-6}Q^{30}. \end{aligned}$$

Suppose next that $p\nmid q$. In this case Lemma 5.8 and the definition of $\Upsilon_p(q,a,b)$ lead to the upper bound

$$\Theta_p(q)\ll \kappa(q)^2P^{-10}S(q).$$

On recalling Lemma 5.7, it follows that the contribution to (5.18) arising from those terms with $p\nmid q$ is of order

$$\begin{aligned} Q^{30}P^{-6}\sum_{M<p\le 2M}\sum_{q=1}^{\infty}\kappa(q)^2q^{-1/3} &\ll MQ^{30}P^{-6}(\log P)^{-1}\prod_{\pi\text{ prime}}(1+19\pi^{-4/3})\\ &\ll MQ^{30}P^{-6}(\log P)^{-1}. \end{aligned}$$

We therefore deduce from (5.18) and the conclusion of the previous paragraph that

$$\mathcal{N}_{0,0}(\mathfrak{M})\ll MQ^{30}P^{-6}(\log P)^{-1},$$

whence by (4.7),

$$\mathcal{N}_{0,0}([0,1)^2)=\mathcal{N}_{0,0}(\mathfrak{M})+\mathcal{N}_{0,0}(\mathfrak{m})\ll MQ^{30}P^{-6}(\log P)^{-1}.$$

This confirms the first of the estimates recorded in (5.3).

Turning our attention next to $\mathcal{N}_{0,0}(\mathfrak{M}\backslash\mathfrak{N})$, we conclude from Lemmata 5.6 and 5.9 that when $\mathfrak{W}=\mathfrak{M}\backslash\mathfrak{N}$, the estimate (5.18) again holds. The analysis of the contribution to (5.18) arising from those terms with $p\|q$ is identical in this case to that above, and so we concentrate on the terms with $p\nmid q$. In such cases, Lemma 5.8 and the definition of $\Upsilon_p(q,a,b)$ establish that

$$\Theta_p(q)\ll \kappa(q)^2P^{-10}Y^{-4/11}S(q),$$

where Y is defined as in the statement of Lemma 5.8. On recalling Lemma 5.7, it now follows that the contribution to (5.18) arising from those terms with $p\nmid q$ is of order

$$\begin{aligned}Q^{30}P^{-6}\sum_{M<p\le 2M}\Big(L^{-4/11}\sum_{q=1}^{\infty}\kappa(q)^2q^{-1/3}+\sum_{q>L}(q/L)^{1/15}\kappa(q)^2q^{-1/3}\Big)&\\ \ll MQ^{30}P^{-6}(\log P)^{-1}L^{-1/15}\prod_{\pi\text{ prime}}(1+19\pi^{-19/15})&\\ \ll MQ^{30}P^{-6}(\log P)^{-1}L^{-1/15}.&\end{aligned}$$

On recalling our comments concerning the terms in (5.18) with $p\|q$, we therefore conclude that

$$\mathcal{N}_{0,0}(\mathfrak{M}\backslash\mathfrak{N})\ll MQ^{30}P^{-6}(\log P)^{-1-\tau},$$

for some positive number τ. This confirms the second of the estimates recorded in (5.3) and completes our pruning operation.

We summarise the deliberations of this section in the form of a lemma.

Lemma 5.10. *For some positive number τ, one has*

$$\mathcal{N}(\mathfrak{n})\ll MP^{-6}Q^{30}(\log P)^{-1-\tau}.$$

Proof. The desired conclusion is immediate from (4.7) and the discussion surrounding (5.3) above, since $\mathfrak{n}$ is the union of $\mathfrak{M}\backslash\mathfrak{N}$ and $\mathfrak{m}$. □

6. A Narrow Set of Major Arcs

Aficionados of the circle method will recognize that the arcs comprising the set $\mathfrak{N}$ are sufficiently few and narrow that an essentially routine analysis will suffice. We begin by recording some notation. Recall the definitions of $S_i(q,a,b)$ and $v_i(\xi,\zeta;B)$ from (5.4) and (5.6). Further, define the functions $W_i(\alpha,\beta)$ for $1\le i\le 4$ and $w_j(\alpha,\beta)$ for $5\le j\le 34$ by taking

$$W_i(\alpha,\beta)=q^{-1}S_i(q,a,b)v_i(\alpha-a/q,\beta-b/q;P) \tag{6.1}$$

and

$$w_j(\alpha,\beta)=(pq)^{-1}S_j(q,a,b)v_j(\alpha-a/q,\beta-b/q;Qp), \tag{6.2}$$

when $(\alpha,\beta)\in\mathfrak{N}(q,a,b)\subseteq\mathfrak{N}$, and otherwise by taking $W_i(\alpha,\beta)=0$ and $w_j(\alpha,\beta)=0$. The functions $W_i(\alpha,\beta)$ and $w_j(\alpha,\beta)$ provide major arc approximations to $F_p(\gamma_i)$ and $f(p^5\gamma_j)$, as the following lemma demonstrates.

Lemma 6.1. *When p is a prime with $M < p \le 2M$, one has*

$$\sup_{(\alpha,\beta)\in\mathfrak{N}} |F_p(\gamma_i) - (1-1/p)W_i(\alpha,\beta)| \ll L^2 \qquad (1 \le i \le 4)$$

and, for a certain positive number c depending at most on η,

$$\sup_{(\alpha,\beta)\in\mathfrak{N}} |f(p^5\gamma_j) - cw_j(\alpha,\beta)| \ll QL^{-10} \qquad (5 \le j \le 34).$$

Proof. When $(\alpha,\beta) \in \mathfrak{N}(q,a,b) \subseteq \mathfrak{N}$ and $M < p \le 2M$, one has $p > M > L \ge q$, so that by a change of variable, the consequent coprimality of p and q ensures that

$$S_i(q, ap^5, bp^5) = S_i(q,a,b) \qquad (1 \le i \le 34), \tag{6.3}$$

and the first conclusion of the lemma now follows instantly from Lemma 5.1 on noting that when $(\alpha,\beta) \in \mathfrak{N}(q,a,b) \subseteq \mathfrak{N}$, one has

$$q^\varepsilon(q + P^5|q\alpha - a| + P^5|q\beta - b|)^{1/2} \ll L^{1+\varepsilon} \ll L^2.$$

Next, from Lemma 8.5 of Wooley [18] (see also Lemma 5.4 of Vaughan [14]) it follows that there exists a positive number c, depending only on η, such that whenever $(\alpha,\beta) \in \mathfrak{N}(q,a,b) \subseteq \mathfrak{N}$, one has

$$\begin{aligned} f(p^5\gamma_j) &= cq^{-1}S_j(q,ap^5,bp^5)v_j(p^5(\alpha - a/q), p^5(\beta - b/q); Q) \\ &\quad + O\left(\frac{Q}{(\log Q)^{1/4}}(q + Q^5p^5|q\alpha - a| + Q^5p^5|q\beta - b|)\right). \end{aligned}$$

By employing (5.7) and (6.3), we find that when $M < p \le 2M$ and $(\alpha,\beta) \in \mathfrak{N}$, one has

$$|f(p^5\gamma_j) - cw_j(\alpha,\beta)| \ll QL^2(\log Q)^{-1/4} \ll QL^{-10}.$$

This completes the proof of the lemma. □

We are now prepared to replace the major arc contribution by the product of a truncated singular series and a truncated singular integral. To this end, we write

$$T(q,a,b) = q^{-34}\prod_{i=1}^{34} S_i(q,a,b), \tag{6.4}$$

$$u_p(\xi,\zeta) = \prod_{i=1}^{4} v_i(\xi,\zeta;P)\prod_{j=5}^{34} v_j(\xi,\zeta;Qp), \tag{6.5}$$

and then define

$$\mathfrak{S}(L) = \sum_{1\le q\le L}\sum_{\substack{a=1\\(q,a,b)=1}}^{q}\sum_{b=1}^{q} T(q,a,b),$$

$$J(L) = \sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} p^{-30}(1-1/p)^4 \iint_{[-LP^{-5},LP^{-5}]^2} u_p(\xi,\zeta)\,d\xi\,d\zeta.$$

Lemma 6.2. *One has*

$$\mathcal{N}(\mathfrak{N}) = c^{30}\mathfrak{S}(L)J(L) + O(MP^{-6}Q^{30}(\log P)^{-1-\tau}),$$

for some positive number τ.

Proof. On making use of the estimates from Lemma 6.1, one finds that whenever $(\alpha,\beta)\in\mathfrak{N}$, one has

$$\left|\prod_{i=1}^{4} F_p(\gamma_i)\prod_{j=5}^{34} f(p^5\gamma_j) - c^{30}(1-1/p)^4\prod_{i=1}^{4} W_i(\alpha,\beta)\prod_{j=5}^{34} w_j(\alpha,\beta)\right| \ll P^4Q^{30}L^{-10}.$$

But on recalling (5.1), one finds that the measure of $\mathfrak{N}$ is $O(L^5P^{-10})$, and hence it follows that

$$\left|\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} \iint_{\mathfrak{N}} \prod_{i=1}^{4} F_p(\gamma_i)\prod_{j=5}^{34} f(p^5\gamma_j)\,d\alpha\,d\beta - c^{30}\mathfrak{S}(L)J(L)\right|$$
$$\ll (L^5P^{-10})(P^4Q^{30}L^{-10})\sum_{M<p\le 2M} 1.$$

The desired conclusion therefore follows from the prime number theorem. □

Before completing the singular series and singular integral to infinity, it is convenient to remark on some simple estimates for $S_i(q,a,b)$ and $v_j(\xi,\zeta;B)$. Observe first that in view of Lemma 3.2 (ii), one may relabel the indices i for $1\le i\le 34$ so that for $1\le i\le 17$ the two coefficient ratios r_{2i-1} and r_{2i} are distinct. But then a change of variables demonstrates that

$$\sum_{\substack{a=1\\ (q,a,b)=1}}^{q}\sum_{b=1}^{q}\left|S_{2i-1}(q,a,b)S_{2i}(q,a,b)\right|^h \le \sum_{\substack{u=1\\ (q,u,v)\le\lambda}}^{q}\sum_{v=1}^{q}\left|S(q,C_iu)S(q,D_iv)\right|^h,$$

where λ, C_i and D_i are positive integers depending only on c_{2i-1}, c_{2i}, d_{2i-1} and d_{2i}, and where $S(q,a)$ is the exponential sum defined in (5.12). But in view of Theorem 4.2 of Vaughan [15], one has

$$S(q,a)\ll q^{4/5}(q,a)^{1/5},$$

so that whenever $(q,u,v)\ll 1$, just as in the proof of Lemma 5.9, one has that

$$S(q,C_iu)S(q,D_iv)\ll q^{8/5}(q,u)^{1/5}(q,v)^{1/5}\ll q^{9/5}. \tag{6.6}$$

Suppose that the indices have been arranged as in the previous paragraph. Then by making a suitable change of variables, and assuming that B and B' are large real numbers with $B\asymp B'$, one finds as in the proof of Lemma 5.8 that

$$\underset{\max\{\xi,\zeta\}\ge H}{\int_0^\infty\int_0^\infty} \left|v_{2i-1}(\xi,\zeta;B)v_{2i}(\xi,\zeta;B')\right|^h d\xi\,d\zeta \le \underset{\max\{\mu,\nu\}\ge\lambda H}{\int_0^\infty\int_0^\infty}\left|v(E_i\mu;B)v(F_i\nu;B')\right|^h d\mu\,d\nu,$$

where λ, E_i and F_i are positive numbers depending only on c_{2i-1}, c_{2i}, d_{2i-1} and d_{2i}, and where we write

$$v(\beta;T)=\int_0^T e(\beta\gamma^5)\,d\gamma.$$

But from Theorem 7.3 of Vaughan [15], for example, one has

$$v(\beta;T)\ll T(1+|\beta|T^5)^{-1/5},$$

and hence

$$|v(E_i\mu;B)v(F_i\nu;B')|\ll BB'(1+|\mu|B^5)^{-1/5}(1+|\nu|(B')^5)^{-1/5}. \tag{6.7}$$

We now complete the truncated singular series $\mathfrak{S}(L)$ to the series

$$\mathfrak{S}=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(q,a,b)=1}}^{q}\sum_{b=1}^{q}T(q,a,b),$$

and extend the truncated singular integral $J(L)$ to the infinite integral

$$J=\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} p^{-30}(1-1/p)^4\iint_{[-\infty,\infty]^2}u_p(\xi,\zeta)\,d\xi\,d\zeta.$$

Lemma 6.3. *One has*

$$\mathfrak{S}-\mathfrak{S}(L)\ll L^{-2/5}\quad\text{and}\quad J-J(L)\ll MP^{-6}Q^{30}(\log P)^{-1}L^{-2}.$$

Proof. Arranging the indices i as in the preamble to this lemma, we obtain by an application of Hölder's inequality the upper bound

$$\begin{aligned}\mathfrak{S}-\mathfrak{S}(L)&=\sum_{q>L}\sum_{\substack{a=1\\(q,a,b)=1}}^{q}\sum_{b=1}^{q}q^{-34}\prod_{i=1}^{34}S_i(q,a,b)\\&\le\sum_{q>L}\prod_{i=1}^{17}\Bigg(\sum_{\substack{a=1\\(q,a,b)=1}}^{q}\sum_{b=1}^{q}q^{-34}\big|S_{2i-1}(q,a,b)S_{2i}(q,a,b)\big|^{17}\Bigg)^{1/17}.\end{aligned}$$

Then in view of the discussion leading to (6.6), we find that

$$\mathfrak{S}-\mathfrak{S}(L)\ll\sum_{q>L}\sum_{a=1}^{q}\sum_{b=1}^{q}q^{-17/5}\ll L^{-2/5},$$

as desired.

Proceeding similarly in our treatment of $J(L)$, we now apply Hölder's inequality in combination with (6.7) to deduce that

$$\begin{aligned} J - J(L) &= \sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} p^{-30}(1-1/p)^4 \underset{\max\{|\xi|,|\zeta|\}\ge LP^{-5}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} \prod_{i=1}^{4} v_i(\xi,\zeta;P)\prod_{j=5}^{34} v_j(\xi,\zeta;Qp)\,d\xi\,d\zeta \\ &\ll M^{-29}(\log P)^{-1} \underset{\max\{\xi,\zeta\}\gg LP^{-5}}{\int_0^{\infty}\int_0^{\infty}} P^{34}(1+\xi P^5)^{-17/5}(1+\zeta P^5)^{-17/5} d\xi\,d\zeta \\ &\ll P^{34}M^{-29}(\log P)^{-1}(P^{-10}L^{-12/5}), \end{aligned}$$

and the desired conclusion follows once again, on recalling that $PM^{-1}=Q$. $\square$

The proof of our main theorem is now rapidly completed. We see from the argument of the proof of Lemma 6.3 that both $\mathfrak{S}$ and J are absolutely convergent. In particular, it follows from the theory familiar to practitioners of the circle method (see for example §2.6 of Vaughan [15], or §10 of Davenport and Lewis [9]) that $\mathfrak{S}$ may be written as an absolutely convergent infinite product $\mathfrak{S}=\prod_p \varpi_p$, where

$$\varpi_p = \lim_{h\to\infty} p^{-32h} M(p^h),$$

and where $M(p^h)$ denotes the number of solutions of the pair of congruences

$$\sum_{i=1}^{34} c_i x_i^5 \equiv \sum_{i=1}^{34} d_i x_i^5 \equiv 0 \pmod{p^h}$$

with $1\le x_i\le p^h$. In view of Lemma 3.2 (iv), one finds via an application of Hensel's Lemma (as in Lemma 6.7 of [18], for example) that $\varpi_p>0$ for all primes p. Moreover, when p is large, the argument of the proof of Lemma 2.12 of [15], together with the discussion leading to (6.6), shows that

$$M(p^h) = p^{32h}(1+O(p^{-7/5})),$$

whence $\varpi_p = 1+O(p^{-7/5})$. Consequently, one may conclude that

$$\mathfrak{S} = \left(\prod_{p\le p_0}\varpi_p\right)\left(\prod_{p>p_0}\varpi_p\right),$$

where p_0 is chosen large enough so that $\varpi_p \ge 1-p^{-6/5}$ for $p>p_0$, and hence

$$\mathfrak{S} \gg \prod_{p>p_0}(1-p^{-6/5}) \gg 1. \tag{6.8}$$

As for the singular integral J, we observe that

$$\iint_{[-\infty,\infty]^2} u_p(\xi,\zeta)\,d\xi\,d\zeta = \iint_{[-\infty,\infty]^2}\int_{\mathcal{D}} e(\xi\mathcal{L}_1(\boldsymbol{\gamma})+\zeta\mathcal{L}_2(\boldsymbol{\gamma}))\,d\boldsymbol{\gamma}\,d\xi\,d\zeta,$$

where we write

$$\mathcal{L}_1(\boldsymbol{\gamma}) = \sum_{i=1}^{34} c_i\gamma_i^5 \quad \text{and} \quad \mathcal{L}_2(\boldsymbol{\gamma}) = \sum_{i=1}^{34} d_i\gamma_i^5,$$

and where $\mathcal{D}$ denotes the box $[0,P]^4\times[0,Qp]^{30}$. Put $\mu = P^5\xi$ and $\nu = P^5\zeta$, and substitute also $\lambda_i = (P^{-1}\gamma_i)^5$ for $1 \le i \le 34$. Then with these changes of variables, we discover that

$$\iint\limits_{[-\infty,\infty]^2} u_p(\xi,\zeta)\,d\xi\,d\zeta = 5^{-34}P^{24}\iint\limits_{[-\infty,\infty]^2}\int\limits_{\mathcal{D}'} \frac{e(\mu L_1(\boldsymbol{\lambda})+\nu L_2(\boldsymbol{\lambda}))}{(\lambda_1\cdots\lambda_{34})^{4/5}}d\boldsymbol{\lambda}\,d\mu\,d\nu,$$

where

$$L_1(\boldsymbol{\lambda}) = c_1\lambda_1+\cdots+c_{34}\lambda_{34} \quad \text{and} \quad L_2(\boldsymbol{\lambda}) = d_1\lambda_1+\cdots+d_{34}\lambda_{34},$$

and where $\mathcal{D}' = [0,1]^4\times[0,(p/M)^5]^{30}$. The equations $L_1(\boldsymbol{\lambda}) = L_2(\boldsymbol{\lambda}) = 0$ define a 32-dimensional linear space, which passes through the point $(\eta_1^5,\ldots,\eta_{34}^5)$. Moreover, Lemma 3.2 (iv) ensures that the latter point lies in the interior of $\mathcal{D}'$. Applying Fourier's integral formula twice, in the shape

$$\lim_{\lambda\to\infty}\int_{-T}^{T}\int_{-\lambda}^{\lambda} V(t)e(t\omega)\,d\omega\,dt = V(0),$$

we therefore obtain

$$\iint\limits_{[-\infty,\infty]^2} u_p(\xi,\zeta)\,d\xi\,d\zeta \gg P^{24}\int\limits_{\substack{\mathcal{D}'\\ L_1(\boldsymbol{\lambda})=L_2(\boldsymbol{\lambda})=0}}(\lambda_1\cdots\lambda_{34})^{-4/5}\,d\lambda_3\cdots d\lambda_{34} \gg P^{24},$$

whence

$$\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} p^{-30}(1-1/p)^4\iint\limits_{[-\infty,\infty]^2} u_p(\xi,\zeta)\,d\xi\,d\zeta \gg P^{24}\sum_{\substack{M<p\le 2M\\ p\equiv -1 \pmod 5}} p^{-30}(1-1/p)^4.$$

The prime number theorem for arithmetic progressions therefore gives the lower bound

$$J \gg M^{-29}(\log P)^{-1}P^{24} = MP^{-6}Q^{30}(\log P)^{-1}. \tag{6.9}$$

On arranging the conclusions of Lemmata 6.2 and 6.3 in concert with the lower bounds (6.8) and (6.9), we finally deduce that

$$\begin{aligned}\mathcal{N}(\mathfrak{N}) &= c^{30}(\mathfrak{S}+O(L^{-2/5}))(J+O(MP^{-6}Q^{30}(\log P)^{-1}L^{-2}))\\ &\qquad + O(MP^{-6}Q^{30}(\log P)^{-1-\tau})\\ &\gg MP^{-6}Q^{30}(\log P)^{-1}.\end{aligned}$$

Thus, in view of the bound recorded in Lemma 5.10, we may conclude that

$$\mathcal{N}([0,1)^2) = \mathcal{N}(\mathfrak{N})+\mathcal{N}(\mathfrak{n}) \gg MP^{-6}Q^{30}(\log P)^{-1}.$$

Since $\mathcal{N}([0,1)^2)\to\infty$ as $P\to\infty$, the conclusion of Theorem 1 follows at last.

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